1. (10%) Use the Laplace transform to solve the system

$$\begin{cases} 2x'-3y'-2y=1 \\ -x'+2y'+3x=0 \end{cases},$$

with initial conditions $x(0)=y(0)=0$, where the prime indicates time derivative.

2. Given a 4x4 matrix **A** as

$$\mathbf{A}=\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \end{bmatrix}$$

(a) (5%) Find the rank of **A**.

(b) (5%) Find the reduced row echelon form of **A**.

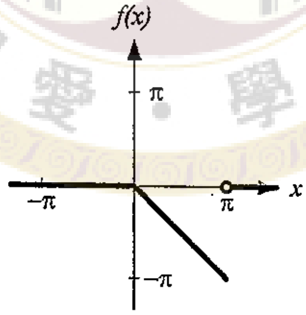
(c) (5%) Let $\mathbf{x}=[x_1,x_2,x_3,x_4]^T$ and $\mathbf{b}=[b_1,b_2,b_3,b_4]^T$, where the superscript *T* denotes the transpose. Assume that the nonhomogeneous system $\mathbf{Ax}=\mathbf{b}$ is consistent and its general solution can be expressed as $\mathbf{x}_g=x_3\mathbf{u}_3+x_4\mathbf{u}_4+\mathbf{x}_p$, where $\mathbf{x}_g$, $\mathbf{u}_3$, $\mathbf{u}_4$, and $\mathbf{x}_p$ are 4×1 matrices. Determine $\mathbf{u}_3$ and $\mathbf{u}_4$.

(d) (5%) Show that $\mathbf{u}_1=[1,1,1,1]^T$ and $\mathbf{u}_2=[1,-1,1,-1]^T$ are eigenvectors of **A** and determine the associated eigenvalues λ_1 and λ_2.

(e) (5%) The characteristic polynomial of **A** can be expressed as $p_A(\lambda)=(\lambda_1-\lambda)^{m_1}(\lambda_2-\lambda)^{m_2}(\lambda_3-\lambda)^{m_3}$, where $|\lambda_1|>|\lambda_2|>|\lambda_3|$. Determine m_3.

(f) (5%) Find a matrix **P** so that $\mathbf{P}^T\mathbf{AP}$ is a diagonal matrix.

3. Function $f(x)$ is shown below.



(a) (6%) Derive *Fourier Cosine Series (FCS)* and *Fourier Since Series (FSS)* representations of $f(x)$ defined on the interval $[0,\pi]$.

(b) (3%) Sketch the *FCS* derived in (a), its first partial sum, and its 10th partial sum with the range of x equal to $[-2\pi,2\pi]$.

(c) (3%) Sketch *Fourier Series (FS)* representation of $f(x)$ defined on the interval $[-\pi,\pi]$, its first partial sum, and its 10th partial sum with the range of x equal to $[-2\pi,2\pi]$.

(d) (3%) Sketch *Fourier Integral* representation of $f(x)$ with the range of x equal to $[-2\pi,2\pi]$.

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4. (15%) Derive solution of the one-dimensional heat equation over the half-line by the Fourier Sine Transform Method. Describe the conditions of $f(x)$ and T so the solution holds.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \quad 0 < x < \infty, t > 0$$
$$u(0,t) = T \quad \text{for } t > 0$$
$$u(x,0) = f(x), \quad \text{for } 0 < x < \infty$$

5. (15%) Write down the answers to the following questions. **(Derivations are not required.)**

(a) Let **v** be a continuous and differentiable vector function, then

$\nabla \times (\nabla \times \mathbf{v}) =$ ________ $- \nabla^2 \mathbf{v}$. Fill in the blank with the correct expression.

(b) Let function $\phi(x,y,z)$ be the solution to the equation $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - 4 = 0$. Evaluate the surface integral $\oiint_S \frac{\partial \phi}{\partial n} dS$ over the surface S (with **n** denoting its unit normal vector) of a sphere of radius 1 centered at the origin.

(c) Let $\mathbf{F} = \left[\frac{x-1}{(x-1)^2 + y^2} - \frac{y}{(x+1)^2 + y^2}\right]\mathbf{i} + \left[\frac{y}{(x-1)^2 + y^2} + \frac{x+1}{(x+1)^2 + y^2}\right]\mathbf{j}$ be a 2-D vector field.

Evaluate the line integral $\oint_C \mathbf{F} \cdot d\mathbf{r}$ over a circle C of radius 2 centered at the origin.

6. (15%) Let $z = x + iy$ denote the complex variable and $f(z)$ a complex function. Answer the following questions. **(Derivations are not required.)**

(a) Let $z_1, z_2, \cdots\cdots, z_n$ be the nth roots of the equation $z^n = 1$ $(n \geq 2)$, find $\sum_{j=1}^{j=n} z_j = ?$

(b) Let $f(z)$ be analytic on the entire domain $|z| < \infty$. If $\oint_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta = z$ over every closed curve C on the complex z-plane, then what is the form of the function $f(z)$?

(c) Evaluate $\oint_C \frac{\cos(z)}{z^4(z+i)} dz$ over a closed contour C defined by $|z| = 1/2$.

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