國立臺灣大學 103 學年度碩士班招生考試試題

2014 NTU MASTER PROGRAM ENTRANCE EXAM
GEOMETRY

1. (25%) Let $\alpha : I \to \mathbb{R}^3$ be a simple closed regular curve with $\kappa \neq 0$. Assume that the unit normal vectors $\mathbf{n} : I \to S^2$ form a simple closed curve. Show that $\mathbf{n}(I)$ separates the sphere into two parts with equal areas.

2. (i) (10%) Show that the only minimal surface of revolution is the catenoid: $\mathbf{y}(u,v) = (a\cosh v\cos u, a\cosh v\sin u, av)$, $u \in (0,,2\pi)$, $v \in (-\infty,\infty)$.

(ii) (10%) Show that the helicoid $\mathbf{x}(u,v) = (a\sinh v\cos u, a\sinh v\sin u, au)$ is a minimal surface, with $\mathbf{y}$ being its conjugate.

(iii) (5%) Construct a family of isometric deformations from $\mathbf{x}$ to $\mathbf{y}$.

3. (i) (10%) Show that

$$K = \frac{1}{(EG-F^2)^2}\left(-\begin{vmatrix} E & F & \frac{1}{2}E_v \\ F & G & \frac{1}{2}G_u \\ \frac{1}{2}E_v & \frac{1}{2}G_u & 0 \end{vmatrix} + \begin{vmatrix} E & F & F_v - \frac{1}{2}G_u \\ F & G & \frac{1}{2}G_v \\ \frac{1}{2}E_u & F_u - \frac{1}{2}E_v & -\frac{1}{2}E_{vv} + F_{uv} - \frac{1}{2}G_{uu} \end{vmatrix}\right).$$

(ii) (10%) We say that the coordinate curves of $\mathbf{x}(u,v)$ form a T-net if the lengths of the opposite sides of any quadrilateral formed by them are equal.
In a T-net, show that we may re-parametrize the coordinates so that $E = G = 1$ and $F = \cos\theta$, where $\theta = \angle(\mathbf{x}_u, \mathbf{x}_v)$, and then

$$K = -\theta_{uv}/\sin\theta.$$

(iii) (5%) Is that possible for a surface S to admit a T-net for all $(u,v) \in \mathbb{R}^2$ so that S has infinite area and $K \leq -c < 0$ for a constant $c > 0$? Explain your answer.

4. (i) (15%) Show that in a geodesic polar coordinate system (ρ,θ) near $p \in S$,

$$E = 1, \quad F = 0, \quad G(p) = 0, \quad \lim_{\rho\to 0}(\sqrt{G})_\rho = 1.$$

(ii) (10%) For $L(r)$ being the length of $\partial B_r(p) \subset S$, show that

$$K(p) = \frac{3}{\pi}\lim_{r\to 0}\frac{2\pi r - L(r)}{r^3}.$$

(You may work on each sub-problem independently. Do give your calculations and proofs in details.)

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