國立臺灣大學 103 學年度碩士班招生考試試題

(1) (10 points) Show that the conjugacy classes of S_n are in one to one correspondence with the partitions of n. (S_n is the permutation group of n elements. Note that every element in S_n has a unique cycle decomposition.)

(2) (15 points) Let $\phi : G \to H$ be a group homomorphism and let N be a normal subgroup of G. Prove or disprove that the image $\phi(N)$ is a normal subgroup of H. (Disproving the statement requires giving an explicit counterexample.)

(3) (15 points) Prove that if $|G| = 462$ then G is not simple. (You need to apply Sylow Theorem.)

(4) (15 points) Let R be a commutative ring. An element $x \in R$ is called nilpotent if $x^n = 0$ for some $n \in \mathbb{Z}^+$. Prove that the set of nilpotent elements, denoted by $\mathcal{N}(R)$, forms an ideal. ($\mathcal{N}(R)$ is called nilradical of R.)

(5) (15 points) Prove that the ideal $\langle 2, x \rangle$ generated by 2 and x in $\mathbb{Z}[x]$ is not principal. ($\mathbb{Z}[x]$ is the ring of polynomials with integral coefficients.)

(6) (15 points) Let p be a prime. Please construct the splitting field $\mathbb{E}$ of $x^p - 2$ over $\mathbb{Q}$ and compute the degree $[\mathbb{E} : \mathbb{Q}]$.

(7) (15 points) Consider the polynomial $f(x) = x^5 - 6x + 3 \in \mathbb{Q}[x]$. First note that $f(x)$ is irreducible due to Eisenstein criterion. Show that the Galois group of $f(x)$ is S_5. (You may apply the Fundamental Theorem of Algebra and Galois theory.)

試題隨卷繳回