1. (10 pts) Let $S_n = \sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}$. Find

$$\lim_{n\to\infty} \frac{S_n}{n^{3/2}}.$$

Justify your answer.

2. (15 pts) Suppose $f : R \to R$ is a bounded continuous function.

(2a) (7 pts) Calculate the following limit

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{\infty} f(t)\frac{\epsilon}{\epsilon^2 + t^2} dt.$$

(2b) (8 pts) Explain clearly why the answer you derived in (2a) is correct.

3. (10 pts) Let $0 < a < b$. Evaluate

$$\lim_{t\to 0}\{\int_0^1 [bx + a(1-x)]^t dx\}^{\frac{1}{t}}.$$

4. (15 pts) Let **S** be the portion of the unit sphere centered at the origin that is cut out by the cone $z \geq \sqrt{x^2+y^2}$. Evaluate $\int\int_{\mathbf{S}} \mathbf{F}\cdot d\mathbf{S}$ where

$$\mathbf{F}(x,y,z) = (xy + \cos z, -yx + x^2 + z^3, 2z^2 + x).$$

5. (10 pts) Let g be the function given by

$$g(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & \text{if } (x,y) \neq (0,0); \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(5a) (5 pts) Calculate the partial derivatives of g at $(0,0)$.

(5b) (5 pts) Show that g is not differentiable at $(0,0)$.

6. (20 pts) Set $I_n(x) = \left[\frac{2}{\pi}\int_{-1}^{x}\sqrt{1-t^2}dt\right]^n$ where $-1 \leq x \leq 1$.

(6a) (5 pts) Evaluate $I_n(x)$ for $-1 \leq x \leq 1$.

(6b) (15 pts) Determine α such that $\lim_{n\to\infty} I_n(1 - t/n^\alpha)$ exists and its limit is not 0 and 1.

7. (20 pts) Set $S(\alpha,\beta) = \sum_{i=1}^n (y_i - \alpha - \beta x_i)^2$ where $\sum_{i=1}^n x_i^2 = n$ and $\sum_{i=1}^n x_i = n/2$. Denote the minimizer of $S(\alpha,\beta)$ by (α_0,β_0).

(7a) (10 pts) Denote the minimizer of $S(\alpha,\beta)$ under the constraint $\alpha^2 + \beta^2 \leq c_1^2$ by (α_2,β_2). Determine (α_2,β_2) in terms of (α_0,β_0) and c_1.

(7b) (10 pts) Denote the minimizer of $S(\alpha,\beta)$ under the constraint $|\alpha| + |\beta| \leq c_2^2$ by (α_1,β_1). Determine (α_1,β_1) in terms of (α_0,β_0) and c_2.

試題隨卷繳回