

第一大題：(50%)

本大題共十小題單選題，考生請作答於電腦閱卷『答案卡』。每小題有 5 個選項，答對得 5 分，答錯或劃記多於一個選項者倒扣 1 分，倒扣到本大題的分數為零分為止。未作答者，不給分亦不扣分。

1. (5%) Let $x, y \in \mathbb{N}$ be two natural numbers. For $n \in \mathbb{N}$, let E_n be the set of pairs of natural numbers (x, y) where n divides both x and y . What is the probability $P[E_n]$ that any randomly chosen natural number is in E_n ?
(A) $1/n$.
(B) $1/n^2$.
(C) $1/n!$.
(D) $1/\log n$.
(E) None of the above.
2. (5%) (Continued from 1.) Given two randomly chosen natural numbers $x, y \in \mathbb{N}$, denote Q as the probability that x and y are relatively prime (i.e. no common positive divisors except 1). Which of the following range does Q lie in?
(A) $[0.0, 0.3)$.
(B) $[0.3, 0.5)$.
(C) $[0.5, 0.7)$.
(D) $[0.7, 0.9)$.
(E) $[0.9, 1.0)$.
3. (5%) Consider an experiment of repeated throwing a fair die. Starting from the first throw, if the die comes up as 6, then the experiment ends (duration of the experiment=0). If, on the other hand, the die comes up as $k, \forall 1 \leq k \leq 5$, then the next throw should wait for k minutes to start. Let T be the expected duration of the experiment rounded to the nearest integer (minutes). What is the sum of all digits in T ? (For example, if $T = 21$, then the sum is 3.)
(A) 3.
(B) 6.
(C) 9.
(D) 10.
(E) None of the above.
4. (5%) Five people get on an elevator on the first floor of an 11-story building. Assume that each person selects a destination floor from 10 possible floors (i.e. from 2 to 11) uniformly at random and

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independently of others. Also, assume that no new people get on the elevator above the first floor. Let S be the expected number of stops this elevator makes above the first floor, and Z be the sum of all digits (including all decimal places) in S . Then which of the following statements is true?

- (A) $15 < Z \leq 19$.
- (B) $19 < Z \leq 22$.
- (C) $22 < Z \leq 24$.
- (D) $24 < Z \leq 25$.
- (E) None of the above.

5. (5%) Assume that the number of typhoons in a given year follows the Poisson distribution with a mean value of 3, and that the occurrence of typhoons in any given year is independent of that in any other years. It can be derived that the probability of having at least 3 typhoons in the 2-year period from 2011 to 2012 can be written as $A+B \cdot e^C$ with A, B, C being constant numbers. What is the value of $A+B+C$?

- (A) 15.
- (B) -20.
- (C) 35.
- (D) -30.
- (E) None of the above.

6. (5%) Let X be the amount of a claim (in 10 thousands) made by a home damaged by the typhoon. Assume that X has the following probability density distribution:

$$f_X(x) = \begin{cases} 3x^{-4}, & x > 1, \\ 0, & \text{otherwise.} \end{cases}$$

Suppose 3 such claims are made during a typhoon attack and claim amounts for individual homes are independent of each other. Let Y be largest of the three claims (in 10 thousands). Then which of the following statements is true?

- (A) The probability that Y is smaller than 1 is 0.2.
 - (B) The probability that Y is smaller than 2 is larger than 0.5.
 - (C) The probability that Y is smaller than 2 is smaller than 0.6.
 - (D) The probability that Y is smaller than 3 is larger than 0.9.
 - (E) None of the above.
7. (5%) (Continued from 6.) Which of the following number is closest to the expected value $E[Y]$ of Y ?
- (A) 1.8.

- (B) 2.5.
- (C) 3.2.
- (D) 4.1.
- (E) 5.4.

8. (5%) Let the moment of a discrete random variable X be given by $E[X^k] = 0.8, \forall k = 1, 2, \dots$. Let $R = P[X = 0]$ be the probability that X equals 0. Then which of the following statements is true?

- (A) $0 \leq R \leq 0.2$.
- (B) $0.2 < R \leq 0.4$.
- (C) $0.4 < R \leq 0.6$.
- (D) $0.6 < R \leq 0.8$.
- (E) $R > 0.8$.

9. (5%) X is a three-dimensional Gaussian random vector with expected value $\mu_X = [4 \ 8 \ 6]'$ and covariance matrix

$$C_X = \begin{bmatrix} 4 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 4 \end{bmatrix}$$

Let D be the maximum number in the correlation matrix R_X . Then which of the following statements is true?

- (A) $20 \leq D < 30$.
- (B) $30 \leq D < 40$.
- (C) $40 \leq D < 50$.
- (D) $50 \leq D < 60$.
- (E) None of the above.

10. (5%) (Continued from 9.) Let random vector $Y = AX + b$, where

$$A = \begin{bmatrix} 1 & 1/2 & 2/3 \\ 1 & -1/2 & 2/3 \end{bmatrix}$$

and $b = [-4 \ -4]'$. Let C_Y be the covariance matrix of Y . Then which of the following statements is true?

- (A) C_Y is a 3×3 matrix.
- (B) The largest number in C_Y is smaller than 12.
- (C) The smallest number in C_Y is larger than 5.
- (D) The sum of all numbers in C_Y is larger than 30.
- (E) None of the above.

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第二大題：(50%)

本大題為計算題，考生請將題號以及詳細的解題過程與結果，清楚地作答於「試卷」之非選擇題作答區。

Consider one specially designed circuit with one capacitor, one inductor, and four resistors. The output voltage of the capacitor is denoted as $x(t)$ and the output current of the inductor is denoted as $y(t)$. Assume that initially $x(0)=0$ and $y(0)=0$ and these two devices might interact with each other by the following behavior. For the capacitor, the changing rate of $x(t)$ is declined at a rate of $-ax(t)$, and simultaneously increased at a rate of $by(t)$, and also positively depends on the independent current source $g(t)=3t$. Similarly, for the inductor, the changing rate of $y(t)$ declines at a rate of $-cy(t)$, and simultaneously increases at a rate of $dx(t)$, and also positively depends on another independent voltage source $h(t)=e^{-t}$. Note that these constants, a, b, c, d , depend on the capacitance of the capacitor, the inductance of the inductor, and the resistance of these resistors. Please answer the following questions:

- (a) Formulate the circuit by a set of first-order differential equations in terms of $x(t)$ and $y(t)$. (5%)
- (b) Formulate the circuit by one second-order differential equation in terms of $y(t)$. (10%)
- (c) Let $a=3, b=1, c=4, d=2$. Use variation of parameters to solve the set of first-order differential equations in (a). That is, find the solution for $x(t)$ and $y(t)$. (15%)
- (d) Let $a=3, b=1, c=4, d=2$. Use the Lapace transform approach to solve the second-order differential equation in (b). That is, find the solution for $y(t)$. (20%)

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