國立臺灣大學 102 學年度碩士班招生考試試題

1 Is the following argument correct or wrong? Why? (10%)

Suppose that $\Re$ is a binary relation on a non-empty set A. If $\Re$ is symmetric and transitive, then $\Re$ is reflexive.

Proof. Let $(x, y) \in \Re$. By the symmetric property, we have $(y, x) \in \Re$. Then, with $(x, y), (y, x) \in \Re$, it follows by the transitive property that we have $(x, x) \in \Re$. As a consequence, $\Re$ is reflexive.

2 Consider ternary strings with symbols 0, 1, 2 used. For $n \geq 1$, let a_n count the number of ternary strings of length n, where there are no consecutive 1's and no consecutive 2's. Show that a_n can be expressed recursively as $2a_{n-1}+a_{n-2}$. (10%)

3 Suppose that G is an undirected simple graph of n vertices. (10%)
(a) Find the number of spanning subgraphs of G that are also induced subgraphs of G.
(b) If every induced subgraph of G is connected, then find the number of edges in G.

4 If G is an undirected simple graph, then there are two vertices in G having equal degree. Why? (10%)

5 Suppose that G is a group, and H, K are two subgroups of G. Prove that if $\gcd(|H|, |K|) = 1$, then $H \cap K = \{e\}$, where e is the identity of G. (10%)

6 If $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ are eigenvalues of matrix A:

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \\ 1 & 4 & 5 & 8 \\ 2 & 3 & 6 & 7 \end{bmatrix}$$

Then $\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_2\lambda_3 + \lambda_2\lambda_4 + \lambda_3\lambda_4 =$________ (5%).

7 If $A = S\Lambda S^{-1}$, then the eigenvalue matrix and eigenvector matrix of $B = \begin{bmatrix} 3A & 0 \\ 0 & 2A \end{bmatrix}$ are

_______ and ________, respectively (5%).

8 Define $T(A) = \frac{A+A^T}{2}$ where A is a $n \times n$ matrix. Then

(a) ker(T)=________ (5%).
(b) (nullity(T), rank(T))=(______,______) (5%).

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9 Suppose that $p_k(x)$ is a polynomial of order k with leading coefficients, a_k, $k = 0, \cdots, n-1$. That is, $p_k(x) = a_k x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$, $k = 0, \cdots, n-1$. Then

$$\begin{vmatrix} p_0(x_1) & p_0(x_2) & \cdots & p_0(x_n) \\ p_1(x_1) & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ p_{n-1}(x_1) & p_{n-1}(x_2) & \cdots & p_{n-1}(x_n) \end{vmatrix} = \_\_\_\_\_\_\_\_\_\_ \text{(10\%)}.$$

10 Let a sequence B_k with $B_o = 0, B_1 = \frac{1}{2}$ and $B_{k+2} = \frac{B_{k+1}+B_k}{2}$, $k = 0, 1, 2 \cdots$. Please find the general expression for B_k=_______ (7%) and $\lim_{k\to\infty} B_k$=_________ (3%).

[True or false] Credits will be given only if all the answers are correct.

11 (5%) Let W_1 and W_2 be subspaces of a vector space V over $\mathbb{R}$.

(a) $W_1 \cap W_2$ is a subspace of V.

(b) $W_1 \cup W_2$ is a subspace of V.

(c) $(V - W_1) \cap W_2$ is a subspace of V.

(d) $V - W_1$ is a subspace of V.

(e) If $W_1 \perp W_2$ then $W_1 = (W_2)^{\perp}$

12 (5%) Suppose that $A, B \in M_{n\times n}$.

(a) A and A^T have the same eigenvalues.

(b) If A is diagonalizable, so is its transpose A^T.

(c) AB and BA have the same eigenvalues.

(d) If α is an eigenvalues of A and β is an eigenvalue of B, then $\alpha\beta$ must be the eigenvalue of AB.

(e) If A and B are both diagonalizable, so is A-B.

試題隨卷繳回