1. **(15%)** Let C be the unit circle $z = e^{i\theta}$ $(-\pi \le \theta \le \pi)$.

   (a) **(7%)** For any real constant r, determine $\int_C \frac{e^{rz}}{z} dz$.

   (b) **(8%)** Apply $z = \cos\theta + i\sin\theta$ and the result of (a), determine $\int_0^{\pi} e^{r\cos\theta} \cos(r\sin\theta) d\theta$.

2. **(20%)** Consider the following boundary value problem:

$$\left(e^{2x}y'\right)' + \lambda e^{2x} y = 0;\quad y(0) = y(\pi) = 0$$

   (a) **(8%)** Determine the eigenvalues and corresponding eigenfunctions.
   (b) **(6%)** Write down the integral form of orthogonal condition of the eigenfunctions.
   (c) **(6%)** Prove the integral form of orthogonal condition.

3. **(20%)** Please solve the following partial differential equation, i.e.,

$$\frac{\partial^2 u}{\partial t^2} + \alpha^2 \frac{\partial^4 u}{\partial x^4} = 0$$

subjected to the boundary conditions of

$$\frac{\partial^2 u(0,t)}{\partial x^2} = \frac{\partial^2 u(L,t)}{\partial x^2} = \frac{\partial^3 u(0,t)}{\partial x^3} = \frac{\partial^3 u(L,t)}{\partial x^3} = 0$$

as well as the initial conditions of

$$u(x,0) = y(x) \text{ and } \frac{\partial u(x,0)}{\partial t} = g(x)$$

for $0 \le x \le L$ and $t \ge 0$.

4. **(15%)** Please evaluate the least squares solution to the following set of systems equations:

$$\begin{cases} x_1 + x_2 = -5 \\ -2x_1 + 3x_2 = 1 \\ -x_2 = 3 \\ 2x_1 + 2x_2 = 2 \\ -3x_1 + 7x_2 = 1 \end{cases}$$

5. **(12%)** Please solve the initial-value problem:

$$\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 10y = 25\cos 4t$$

subjected to the initial conditions of

$$y(0) = \frac{1}{2},\quad y'(0) = 0$$

What is the physical meaning of the homogeneous solution and particular solution, respectively? Plot the individual graphs of these two solutions first and then their combined solution.

6. **(18%)** Please solve the following equation by applying steps (a) – (c):

$$t^2 y'' + t y' + (t^2 - n^2) y = 0$$

   (a) **(6%)** Rewrite this equation using the Change of Variables by setting $y(t) = t^{-n} w(t)$.
   (b) **(6%)** Apply the Laplace transform to obtain *W(s)*.
   (c) **(6%)** Use the following binomial series to expand *W(s)* and then the solution *y(t)* for the first four terms.

$$(1+x)^k = \sum_{m=0}^{\infty} \binom{k}{m} x^m \quad for\ |x| < 1 \qquad \text{where} \quad \binom{k}{m} = \begin{cases} 1 & for\ m = 0 \\ \frac{k(k-1)\cdots(k-m+1)}{m!} & for\ m = 1, 2, 3, \cdots \end{cases}$$

試題隨卷繳回