# 國立臺灣大學 102 學年度碩士班招生考試試題



1.(50%) Let us consider the following linear system of ordinary differential equations (ODEs):

$$\frac{d}{dt}\begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{A}\begin{bmatrix} x \\ y \end{bmatrix}, \quad x(0) = x_0, \quad y(0) = y_0, \tag{1}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{2}$$

is a constant matrix and $(x_0, y_0) \neq (0, 0)$.

(a)(9%) Write the determinant $q = \det(\mathbf{A})$, and under the values of $q$ discuss the null space (kernel space) and column space and write the inverse matrix $\mathbf{A}^{-1}$.

(b)(6%) Let $\mathbf{a}_1 = (a_{11}, a_{12})$, $\mathbf{a}_2 = (a_{21}, a_{22})$ with $\mathbf{a}_1 \cdot \mathbf{a}_2 \neq 0$. Make a pair of orthonormal vectors $\mathbf{u}$ and $\mathbf{v}$ by using the Gram-Schmidt process.

(c)(7%) Derive the eigenvalues of A in Eq. (2) in terms of $p = \mathrm{tr}(\mathbf{A})$ and $q = \det(\mathbf{A})$. Under what condition the eigenvalues are real?

(d)(6%) What is the equilibrium point of Eq. (1), and under what conditions about $p$ and $q$ the equilibrium point is unstable, neutral stable and stable?

(e)(6%) For Eq. (1) with $a_{11} = 0$, $a_{12} = 1$, $a_{21} = -k/m$ and $a_{22} = -c/m$ as a mechanical free vibration system with mass $m > 0$, spring constant $k$ and damping constant $c$, when the system is over-damped, critically-damped and under-damped.

(f)(6%) In terms of $k$, $c$ and $D = c^2 - 4mk$, write the conditions for (i) saddle point, (ii) unstable node, and (iii) stable focus.

(g)(10%) In order to keep the length $x^2(t) + y^2(t)$ invariant, the matrix $\mathbf{A}$ in Eqs. (1) and (2) must satisfy what condition? For this case derive the solutions of $x(t)$ and $y(t)$ in terms of initial conditions $(x_0, y_0)$ and prove $x^2(t) + y^2(t) = x_0^2 + y_0^2$.

2.(20%) Consider the following Sturm-Liouville boundary value problem:

$$y''(x) + \lambda^2 y(x) = \lambda^2\delta + c_0, \quad 0 < x < 1,$$
$$y(0) = 0, \quad y'(0) = 0, \quad y(1) = \delta, \quad y'(1) = 0,$$

where $c_0 \neq 0$ and $\delta \neq 0$ are unknown constants. (a)(15%) Derive the general solution and determine the eigenvalues $\lambda$ and eigenfunctions $y(x)$. (b)(5%) Write the relation between $c_0$, $\delta$ and $\lambda$.

3.(15%) Consider the following partial differential equation (PDE):

$$u_t(x,t) = u_{xx}(x,t), \quad -\pi < x < \pi, \ t > 0,$$
$$u(-\pi, t) = u(\pi, t) = 0, \quad u(x, 0) = f(x) = x + x^2.$$

By using the separation of variable and Fourier series expansion method derive the solution of the above PDE.

4.(15%) Consider the following PDE:

$$u_{tt}(x,t) = u_{xx}(x,t) + \cos(\omega t), \quad 0 < x < \infty, \ t > 0$$
$$u(x,0) = u_t(x,0) = 0, \quad u(0,t) = 0, \quad u_x(x,t) \to 0, \quad \text{as } x \to \infty,$$

where $\omega$ is a constant. By using the Laplace transform method derive the solution of the above PDE.

試題隨卷繳回