There are five problems 1 ~ 5 in total; some problems contain sub-problems, indexed by (a), (b), etc.

1. [15%] Let V be a finite dimensional vector space over a field F and $T : V \to V$ a linear operator. Let W be a subspace of V such that $T(W) \subset W$. Suppose that $v_1, v_2, \cdots, v_r$ are eigenvectors of T associated with distinct eigenvalues such that $v_1 + v_2 + \cdots + v_r \in W$. Show that $v_i \in W$ for all $i = 1, 2, \cdots, r$.

2. Let A and B be two $n \times n$ matrices over a field F.
   (a) [20%] Show that AB and BA have the same trace and the same determinant.
   (b) [10%] Give an explicit example of A and B such that AB and BA have different minimal polynomials. Remember to verify your answer.

3. [10%] Let V be a finite dimensional vector space over a field F and V^* be the dual space (i.e., V^* = all linear maps from V to F). Show that two non-zero vectors $v, w \in W$ are linearly independent if and only if there exists an $f \in V^*$ such that $f(v) = 0, f(w) \neq 0$.

4. Consider the space $M_n(F)$ of $n \times n$ matrices over a field F. Two matrices $A, B \in M_n(F)$ are called *similar* if there exists an invertible matrix $Q \in M_n(F)$ such that $A = Q^{-1}BQ$. In this case A and B have the same characteristic polynomial. A *conjugacy class* $\mathcal{C}$ is a maximal subset $\mathcal{C}$ of $M_n(F)$ such that all $A, B \in \mathcal{C}$ are similar. In other words, the conjugacy class containing A is the set

$$\{B \in M_n(F) \mid A \text{ and } B \text{ are similar}\}.$$

   The *characteristic polynomial* of a conjugacy class $\mathcal{C}$ is defined to be the characteristic polynomial of a matrix A in $\mathcal{C}$.
   (a) [15%] In the case $n = 12$ and $F = \mathbb{C}$, the field of complex numbers, what is the number of conjugacy classes with characteristic polynomial $(x^3 - 1)^4$? Verify your answer.
   (b) [15%] In the case $n = 12$ and $F = \mathbb{R}$, the field of real numbers, what is the number of conjugacy classes with characteristic polynomial $(x^3 - 1)^4$? Verify your answer.

5. [15%] Let V be a finite dimensional vector space over $\mathbb{R}$ with an inner product $(\,,)$. Show that for any linear $T : V \to \mathbb{R}$, there exists a unique $v_T \in V$ such that $T(x) = (v_T, x)$ for all $x \in V$. Also show that the map from the dual V^* (= the vector space of all linear maps from V to $\mathbb{R}$) to V assigning each T to v_T is linear and is an isomorphism.

試題隨卷繳回