國立臺灣大學101學年度碩士班招生考試試題

1. (14%) Find the general solutions of the following ODE's.
   (a) $y'' + 2y' + y = e^{-x}$
   (b) $x^2y'' - xy' + y = 0$

2. (14%)
   (a) Find the inverse Laplace transform $y(t)$
   $$Y(s) = \frac{s^3}{s^4 + 4}$$
   (b) Find the Laplace transform Y(s)
   $$y(t) = e^t, \quad 0 \le t < 5$$
   $$= e^{10-t}, \quad 5 \le t$$

3. (15%) The definition of Bessel function of order n is
   $$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!}$$
   Show that
   $$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)\text{ , and}$$
   $$\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x).$$

4. (7%) With Bessel function $J_{1/2}(x) = \sqrt{2/\pi x}\sin x$, determine the Bessel function $J_{-1/2}(x)$.

5. (10%) Find the Fourier series of the following function $f(x)$.
   $$f(x) = \begin{cases} x - \pi & \text{if } 0 < x < \pi \\ -\pi & \text{if } \pi < x < 2\pi \end{cases} \quad \text{and} \quad f(x + 2\pi) = f(x)$$

6. (10%) A field vector $\underline{F} = 3xy\mathbf{i} + 7y^2\mathbf{j} + z^2\mathbf{k}$ is given. A three dimensional region is bounded by a closed surface S consisting of a cylinder surface $x^2+y^2=1$ (at $0<z<1$) and two circular disks $z=0$ and $z=1$ (at $x^2+y^2\le 1$). $\underline{n}$ is the unit normal vector of the surface S.
   (a) Please find $\underline{F} \bullet \underline{n}$ at the point $(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, \frac{1}{2})$?
   (b) Use divergence theorem of Gauss to find the value of $\iint_S \underline{F} \bullet \underline{n}\, dA$

7. (15%) Please use separation of variables solving the following partial differential equation.
   $$\frac{\partial^2 u(x,t)}{\partial t^2} = 9\frac{\partial^2 u(x,t)}{\partial x^2} \quad \text{for } 0 \le x \le 1 \text{ and } t \ge 0$$

見背面

$u(0,t)=u(1,t)=0 \quad \text{for } t \ge 0$

$\frac{\partial u}{\partial x}(x,0)=\cos 3\pi x \quad \text{for } 0 \le x \le 1$

$\frac{\partial u}{\partial t}(x,0)=\sin \pi x \quad \text{for } 0 \le x \le 1$

8. (15%)

(a) Please find the eigenvalues and eigenvectors of a matrix A

$$\underline{A}=\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

(b) Set $\underline{X}=[\underline{x}_1, \underline{x}_2]$, which is the matrix with a basis of eigenvectors of $\underline{A}$ ($\underline{x}_1$ and $\underline{x}_2$) as column vectors. What is its inverse ($\underline{X}^{-1}$)?

(c)Please solve the following differential equations. Both y_1 and y_2 are only functions of t.

$$y_1' = 2y_1 + 2y_2$$

$$y_2' = y_1 + 3y_2 + \frac{3}{2}$$

The initial conditions are

$$y_1(t=0)=0$$

$$y_2(t=0)=-\frac{3}{4}$$

(Hint: you may set $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \underline{X}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ and try to solve z first. In addition, $\underline{X^{-1}AX}=\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, where λ_1, λ_2 are eigenvalues of A.)

試題隨卷繳回