1. (10%) Suppose that U, V and W are independent random variables (not necessarily identically distributed) with $Var(U) = Var(V) = Var(W) = 1$. Let $X = U + V$ and $Y = W + V$. Find $Cov(X, Y)$ and the correlation coefficient of X and Y.

2. (20%) Let X have a binomial $Bin(2, 1/2)$ distribution. (Note that $P(X = 0) = 1/4$.) Conditional on $X = x$, the random variable Y has a Poisson distribution with parameter $\lambda(1 + x)$ and

$$P(Y = y|X = x) = \exp(-\lambda(1 + x))\frac{[\lambda(1 + x)]^y}{y!},$$

for $x = 0, 1, 2$, and $y = 0, 1, \ldots$.

(a) (10%) Calculate $P(Y = 2)$ and $E(Y)$.

(b) (10%) Calculate $Var(Y)$.

3. (15%) Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with mean 2 and variance 4, and let $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Let $Z_n = 1/\bar{X}_n$. Find the limiting distribution of Z_n. (i.e., Determine a_n and b such that $a_n(Z_n - b)$ converges in distribution to a non-degenerate distribution.)

4. (25%) Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed with probability density function

$$f(x|\theta) = \theta x^{\theta-1}, \quad 0 \leq x \leq 1,$$

where the parameter $\theta > 0$.

(a) (5%) Derive the log likelihood function and show that it depends on the data $x_, \ldots, x_n$ only through $\sum_{i=1}^{n} \log x_i$.

(b) (5%) Derive the maximum likelihood estimator for θ.

(c) (10%) Derive the method of moments estimator for θ.

(d) (5%) Show that either estimators you derived in (b) or (c) is consistent or not consistent.

5. (10%) X_1 and X_2 are independent normally distributed random variables with mean μ and variances σ_1^2 and σ_2^2 respectively. The variances are known and we are interested in estimating the mean μ. Consider estimators of the form $W_{a,b} = aX_1 + bX_2$. Find the minimum variance unbiased estimator in this class of estimators.

6. (20%) Consider the following probability mass function:

$$f(x|\theta) = \begin{cases} 0.4 - \theta, & x = 1 \\ 0.3, & x = 2 \\ 0.1 + \theta, & x = 3 \\ 0.2, & x = 4 \\ 0, & \text{otherwise} \end{cases}$$

where $-0.1 \leq \theta \leq 0.4$. Suppose X is a random variable with this pmf.

(a) (10%) Suppose we have a random sample of size 1 from $f(x|\theta)$. Give the critical region for a uniformly most powerful test (with level 0.1) of $H_0 : \theta = 0$ versus. $H_1 : \theta = 0.2$.

(b) (10%) Would the critical region be the same for a uniformly most powerful test (with level 0.1) of $H_0 : \theta = 0$ versus $H_1 : \theta > 0$? Justify your answer.

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