※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

You should include in your answer every piece of computation and every piece of reasoning so that the corresponding partial credit could be gained.

1. (20%) For nay two vectors $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ in $\mathbb{R}^n$, The *convolution* of $\mathbf{x}$ and $\mathbf{y}$ is the vector $\mathbf{x} * \mathbf{y}$ in $\mathbb{R}^n$ with

$$(\mathbf{x} * \mathbf{y})_i = x_1 y_i + x_2 y_{i-1} + \ldots + x_i y_1 + x_{i+1} y_n + x_{i+2} y_{n-1} + \ldots + x_n y_{i+1}$$

for $1 \leq i \leq n$. For $n = 2$, this gives simply $(x_1, x_2) * (y_1, y_2) = (x_1 y_1 + x_2 y_2, x_1 y_2 + x_2 y_1)$.

(a) Find the explicit formula for $(x_1, x_2, x_3) * (y_1, y_2, y_3)$.

(b) Compute $(3, -1, 4, 1) * (5, -9, 2, 6)$.

(c) Prove that $*$ is commutative for general n.

(d) Find an "identity" $\mathbf{e}$ in $\mathbb{R}^n$ such that $\mathbf{e} * \mathbf{x} = \mathbf{x} * \mathbf{e} = \mathbf{x}$ for all $\mathbf{x}$ in $\mathbb{R}^n$. Prove that the identity is unique.

2. (20%) Denote $\mathbb{Z}_5$ the finite field of 5 elements, in which the addition/muplitiplication are the same as in integers except taking modulo 5. Notice that there are exactly 625 2×2 matrices whose entries are in $\mathbb{Z}_5$.

(a) Determine the total number of 2×2 symmetric matrices over $\mathbb{Z}_5$.

(b) Determine the total number of nonsingular 2×2 symmetric matrices over $\mathbb{Z}_5$.

(c) Determine the total number of nonsingular 2×2 matrices over $\mathbb{Z}_5$.

(d) For every $n \geq 1$, determine the total number of upper-triangular nonsingular $n \times n$ matrices over $\mathbb{Z}_5$.

3. (20%) The *rank* $\operatorname{rank}(A)$ of a matrix A is the maximum number of linearly independent rows in A.

(a) Prove that for any $m \times n$ matrix A and any $n \times p$ matrix B, we have $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.

(b) Prove that for any $n \times n$ matrix A and any $n \times p$ matrix B, if A is invertible then $\operatorname{rank}(AB) = \operatorname{rank}(B)$.

(c) Determine $\operatorname{rank}(I_n), \operatorname{rank}(J_n)$ and $\operatorname{rank}(J_n - I_n)$, where I_n is the $n \times n$ identity matrix and J_n is the $n \times n$ matrix whose entries are all 1.

(d) Suppose $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k, \mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$ are $2k$ vectors in $\mathbb{R}^n$ such that $\mathbf{x}_i \mathbf{y}_i^{\mathrm{T}} = 0$ for $1 \leq i \leq k$ and $\mathbf{x}_i \mathbf{y}_j^{\mathrm{T}} = 1$ for $1 \leq i < j \leq k$. Prove that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ (also $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_k$) are linearly independent. Consequently, $k \leq n$.

4. (20%) Suppose M is an $n \times n$ matrix in which all entries are nonnegative and all column sums are 1.

(a) Prove that 1 is an eigenvalue of M.

(b) Prove that for each eigenvalue λ of M, we have $|\lambda| \leq 1$.

(c) For every n, give an $n \times n$ matrix whose eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are all distinct but $|\lambda_i| = 1$ for all i.

(d) Prove that if all entries of M are positive, then M has exactly one eigenvalue λ with $|\lambda| = 1$.

5. (20%) (a) Prove that for any $n \times n$ matrix A the series $\sum_{k=0}^{\infty} \frac{A^k}{k!}$ converges; the sum is denoted by e^A.

(b) Compute e^{tA} for $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

(c) Prove that if λ is an eigenvalue of A then e^{λ} is an eigenvalue of e^A.

(d) Prove that $\frac{d}{dt} e^{tA} = A e^{tA}$.