國立臺灣大學101學年度碩士班招生考試試題

1. Solve the ODEs and find its maximal interval of existence
   (a) (5%) $y' = 1 + y^2,\ y(0) = y_0$;
   (b) (5%) $y' = y \ln y,\ y(0) = y_0 > 0$.

2. (a) (5%) Find the general solutions of the ODE: $y' + \frac{1}{t}y = t^2$.
   (b) (5%) Find the Laplace transform of the function $\cos \omega t$.

3. (10%) Find general solutions of the ODE: $x^2y'' + bxy' + cy = 0$, where b, c are constants.

4. Consider the logistic model

$$y' = ry\left(1 - \frac{y}{K}\right),\ y(0) = y_0,$$

   where $r > 0, K > 0$ are two constants and $0 < y_0 < K$.
   (a) (5%) Find its general solutions.
   (b) (5%) Discuss the solution behavior (stable, unstable) as t tends to infinity.
   (c) (10%) Consider the harvest model

$$y' = ry\left(1 - \frac{y}{K}\right) - ey,\ 0 < y(0) = y_0 < K,$$

   where $e > 0$ is the harvest rate. Discuss how the asymptotic solution depends on the harvest rate e.

5. Consider the damped oscillation system with periodic forcing:

$$y'' + \alpha y' + \beta y = F_0 \cos(\Omega t),$$

   where, $\alpha > 0, \beta > 0$, F_0 and Ω are constants.
   (a) (10%) Find the solution to this system with initial condition

$$y(0) = y_0,\ y'(0) = v_0.$$

   (b) (10%) Discuss the asymptotic behaviors of the solutions (that is, what is the limit of $y(t)$ as $t \to \infty$).

6. Consider the conservative mechanical system in $\mathbb{R}$ with a unit mass:

$$\ddot{x} = -V'(x),$$

   where $V(x)$ is the potential function and $-V'(x)$ is the force.
   (a) (10%) Show that the energy $E(t) := \frac{1}{2}|\dot{x}(t)|^2 + V(x(t))$ is independent of time.
   (b) (10%) Show that if $V(x) \to \infty$ as $|x| \to \infty$, then all solutions with finite energy are periodic. (Constant solution is treated as a periodic solution.)
   (c) (10%) Suppose the system has a damping term:

$$\ddot{x} = -V'(x) - \beta\dot{x}$$

   where $\beta > 0$ is the damping coefficient. Assume $V(x)$ is strictly convex and $V(x) \to \infty$ as $|x| \to \infty$. Show that all solutions tend to x_0, the unique global minimum of V.

試題隨卷繳回