1. (30%). Suppose $\mathbf{A}$ is a real symmetric square matrix such that $\mathbf{A}^2 = \mathbf{A}$.

(a). (5%). Find the determinant of $\mathbf{A}$ (i.e.; det $\mathbf{A}$).

(b). (5%). Fnd the eigenvalues of $\mathbf{A}$.

(c). (5%). Let $\mathbf{x}$ be a column vector and the symbol $|\mathbf{x}|$ denote its vector length.

Find $(\mathbf{Ax})^T(\mathbf{x}-\mathbf{Ax})$ (i.e.; the inner product of two column vectors $\mathbf{Ax}$ and $\mathbf{x}$-$\mathbf{Ax}$)

(d). (5%). Show that $|\mathbf{Ax}| \le 1$ for any column vector $\mathbf{x}$ with $|\mathbf{x}| = 1$.

(e). (10%). Suppose $\mathbf{A}$ here denote a real symmetric $3\times3$ matrix. Let $\mathbf{t}$, $\mathbf{m}$ and $\mathbf{n}$ be its orthonormal eigenvectors such that

$$\mathbf{Am}=\mathbf{m},\quad \mathbf{An}=\mathbf{n},\quad \mathbf{At}=0,$$
$$|\mathbf{m}|=1,\quad |\mathbf{n}|=1,\quad |\mathbf{t}|=1,\quad \mathbf{t}^T\mathbf{m}=\mathbf{t}^T\mathbf{n}=\mathbf{m}^T\mathbf{n}=0$$

Let $\mathbf{P}=(\mathbf{m}|\mathbf{n}|\mathbf{t})$ and $\Lambda=\begin{pmatrix}1&0&0\\0&1&0\\0&0&0\end{pmatrix}$ be the $3\times3$ matrices containing the eigenvectors and eigenvalues of $\mathbf{A}$. Suppose $\mathbf{t}=(t_1,t_2,t_3)^T$. Find $\mathbf{PAP}^{-1}$ in terms of $t_1, t_2$ and $t_3$.

2. (a) (6%) Given a 1st order system ODE:

$$\mathbf{y}'(x)=A\mathbf{y}(x),\ x>0$$
$$\mathbf{y}(0)=\begin{Bmatrix}-1\\1\end{Bmatrix},\ \text{where } A=\begin{bmatrix}2&-1\\0&2\end{bmatrix}.$$

Solve $\mathbf{y}(x)$.

(b) (6%) Given a 1st order ODE:

$$y'(x)=\frac{y\ln y}{x},\ y(0)=-1.$$

Solve $y(x)$.

(c) (8%)

Find the general solution of the following 2nd order ODE:

$$x^2y''-xy'+y=0.$$

(d) (10%) Given a 2nd order ODE:

$$y''(x)+k^2y(x)=f(x),\ x>0,$$

where $k$ is a real number, and $f(x)$ is an arbitrary piecewisely continuous real function.

Solve $y(x)$.

3. The height of a hill (in meter) is given by

$$h(x,y) = 10\left(2xy - 3x^2 - 4y^2 - 18x + 28y + 12\right),$$

where $y$ is the distance (in km) north, $x$ the distance east of Heaven city.

(a) (3%) Where is the top of the hill located?

(b) (2%) How high is the hill?

(c) (3%) How steep is the slope at a point 1 km north and 1 km east of Heaven city, and in what direction?

(d) (2%) Calculate $\nabla \cdot \nabla h$ and $\nabla \times \nabla h$.

4. (10%) Evaluate the integral $\oint_C \left[2xy\,dx + \left(e^x + x^2\right)dy\right]$ along the curve $C$, where $C$ is the boundary of the triangle with vertices (0,0), (1,0) and (1,1), along clockwise direction.

5. The one dimensional wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

describes the transverse displacement, $u$ $(= u(x,t))$, of an elastic stretched string, where $c = \sqrt{T/\rho}$ is the wave speed, with $T$ the tension in the string, and $\rho$ the density of the string.

(a) (15%) Solve the equation subject to boundary conditions $u(0,t) = u(l,t) = 0$ and initial conditions $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$. Here $l$ is the length of the string.

(b) (5%) Describe the physics of the problem, including the equation, the boundary conditions, the initial conditions and the solution. State also the assumptions for deriving the wave equation.