1. (15%)

   (1) Solve $x^3 y''' + x y' - y = 0$.

   (2) Solve $(y - x\sqrt{x^2+y^2})\,dx + (x - y\sqrt{x^2+y^2})\,dy = 0$.

2. (15%) Knowing $L[J_o(t)] = 1/\sqrt{s^2+1}$ and $y(0)=1$, $y'(0)=0$;

   use Laplace transform to solve $t y'' + y' + t y = 0$.

3. (10%)

   (1) Write the 3 x 3 matrix of the geometric transformation representing the z-axis counterclockwise rotation (i.e., the axis of rotation perpendicular to the x-y plane).

   (2) Find the eigenvalues and eigenvectors of this 3 x 3 transformation matrix.

4. Consider an elastic string of length L, fixed at its ends on the x axis at x=0 and x=L. Its displacement function satisfies:

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for } 0 < x < L,\ t > 0,$$

$$y(x,0) = f(x) \quad \text{for } 0 \le x \le L,$$

$$\frac{\partial y}{\partial t}(x,0) = g(x) \quad \text{for } 0 \le x \le L.$$

   (1) (8%) For zero initial velocity, which of the functions listed below gives the correct displacement? (Justification of your answer is required to get credit.)

   (a) $y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$

   (b) $y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi ct}{L}\right)$

   (c) $y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\cos\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$

   (d) $y(x,t) = \frac{2}{L}\sum_{n=1}^{\infty}\left(\int_0^L f(\xi)\sin\left(\frac{n\pi\xi}{L}\right)d\xi\right)\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)$

   (2) (6%) For zero initial velocity and $f(x) = 95\sin\left(\frac{6\pi x}{L}\right)$, determine the displacement function.

   (3) (6%) For zero initial displacement and $g(x) = 95\sin\left(\frac{6\pi x}{L}\right)$, the displacement is

   $y(x,t) = \frac{95}{3\pi c}\sin\left(\frac{3\pi x}{L}\right)\sin\left(\frac{3\pi ct}{L}\right)$. What is the displacement if $f(x) = g(x) = 95\sin\left(\frac{6\pi x}{L}\right)$?

   (4) (5%) For zero initial velocity and $f(x) = H(x-\frac{1}{3}L) - H(x-\frac{2}{3}L)$, sketch $y(x,\frac{L}{6c})$.

   (5) (5%) Following (4), sketch $y(x,\frac{L}{2c})$

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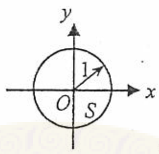
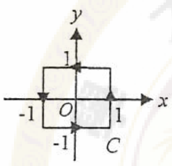
5. (15%) Write down the answers to the following questions. **(Derivations are not required.)**

(1) Let $\Psi(r,\theta)=\cos\theta/r^2$. Evaluate the line integral $\oint_C \Psi\,\underline{\mathbf{n}}\,d\ell$ over a circle C of radius 2 centered at the origin. ($\underline{\mathbf{n}}$ denotes unit vector normal to the contour of the circle C.)

(2) Let $\Phi(r,\theta)$ satisfy the 2-D Laplace equation (i.e. $\nabla^2\Phi=0$); and $\Phi=\cos^2\theta-\sin^2\theta$, $\partial\Phi/\partial r=2\cos(2\theta)$ along the contour of a unit circular disk S. Evaluate the surface integral $\iint_S \nabla\Phi\cdot\nabla\Phi\,dA$ over the circular disk S.



(3) Let $\underline{\mathbf{F}}=\dfrac{x}{x^2+y^2}\,\underline{\mathbf{i}}+\dfrac{y}{x^2+y^2}\,\underline{\mathbf{j}}$ be a 2-D vector field. Evaluate $\oint_C \underline{\mathbf{F}}\cdot d\underline{\mathbf{r}}$ over a closed contour C given by:



6. (15%) Let $z=x+iy$ denote the complex variable, $\bar{z}=x-iy$ the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. **(Derivations are not required.)**

(1) Evaluate $\oint_C \dfrac{\bar{z}}{\bar{z}-(i/2)}\,d\bar{z}$ over C: $|z|=1$.

(2) Let $f(z)$ be analytic on the upper-half of z-plane and $|f(z)|\to 0$ as $|z|\to\infty$. If on the real axis, $f(z)$ takes the form $\dfrac{x-i}{x^2+1}$, find the function $f(z)$.

<hint>: Apply Cauchy integral formula.

(3) If the Laurent series expansion of $f(z)=\dfrac{z}{(z-i)(z+1)^2}$ about $z=-1$ is denoted by $\sum_{n=-\infty}^{n=+\infty} a_n(z+1)^n$, find $\sum_{n=-\infty}^{n=+\infty} a_n = ?$