1.(20%)
Give the following matrix:

$$\mathbf{A} = \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

(a)(5%) Find the eigenvalues of $\mathbf{A}$.
(b)(5%) Find the eigenvectors of $\mathbf{A}$.
(c)(5%) Diagonalize $\mathbf{A}$, i.e., find matrices $\mathbf{P}$ and $\mathbf{D}$ such that $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D}$, where $\mathbf{D}$ is a diagonal matrix.
(d)(5%) Find $\mathbf{A}^6$ [By using the results from (c)].

2.(15%) Let $\mathcal{L}[f(t)] = \int_0^\infty f(t)e^{-st}dt$ be the Laplace transformation of $f(t)$.
(a)(5%) Show that $\mathcal{L}[t] = 1/s^2$.
(b)(5%) Show that $\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0)$.
(c)(5%) Find $\mathcal{L}[\cos(\omega t + \theta)]$, where ω and θ are constants.

3.(15%) Given a vector $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} + z^2\mathbf{k}$, and a cylindrical surface S of $x^2 + y^2 \leq 4$, $0 \leq z \leq 5$. Using the Gauss divergence theorem, evaluate the surface integral.

4(a)(10%) Find the general solution of

$$\frac{dy(x)}{dx} = x^3(y-x)^2 + x^{-1}y, \quad x > 0.$$

4(b)(5%) If we further require $y(1) = -2$, what is the solution of $y(x)$.

5(a)(5%) Find the Fourier series expansion for the following function

$$f(x) = x + x^2, \quad -\pi < x < \pi.$$

5(b)(3%) Express the Parseval's relation in terms of the Fourier coefficients a_0, a_k, b_k, $k = 1, 2, \cdots$.
5(c)(7%) Can you prove that the Fourier coefficients in (a) satisfy the Parseval's relation. (Hint $\sum_{k=1}^{\infty} 1/k^4 = \pi^4/90$ and $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$.)

6.(20%) For the one-dimensional heat conduction equation:

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

the solution with $u(x,t) = q(x-ct)$ is called a wave solution, where $c \in \mathbb{R}$ is the wave propagation speed. Let $\xi = x - ct$, and $q' = dq/d\xi$ and $q'' = d^2q/d\xi^2$.
(a)(5%) Derive an ordinary differential equation for $q(\xi)$, that Eq. (1) has a wave solution $u = q(x-ct)$.
(b)(15%) Let

$$f(u) = u(1-u)(u-0.3). \tag{2}$$

Derive the wave solution $u(x-ct)$ of Eq. (1) with the above $f(u)$ [Hint $q' = kq(q-1)$, and c and k are to be determined]. Write $k =?$ and $c =?$.