(1) (15%) Find all possible $(w,x,y,z)\in\mathbb{R}^4$ satisfying

$$\begin{aligned} w+2x+2z&=27\\ 3w+5x-y+6z&=65\\ 2w+4x+y+2z&=53\\ 2w-7y+11z&=-3 \end{aligned}$$

(2) (20%)

(a) If A and B are 3×3 *real* matrices such that $AB=-BA$, then at least of one of A and B is not invertible. Prove or disprove this statement.

(b) If A and B are 4×4 *real* matrices such that $AB=-BA$, then at least of one of A and B is not invertible. Prove or disprove this statement.

(3)(15%) Consider the vector space of polynomials of degree at most 7:

$$V=\{a_0+a_1x+\cdots+a_6x^6+a_7x^7 \mid a_0,\cdots,a_7\in\mathbb{R}\}.$$

Show that for each $g(x)\in V$, there exist ***unique*** $a_0,a_1,\cdots,a_7\in\mathbb{R}$ so that the equality

$$\sum_{i=0}^{7}a_i f(i)=\int_0^1 f(x)g(x)dx$$

holds for all $f(x)\in V$, and conversely, for every $a_0,a_1,\cdots,a_7\in\mathbb{R}$, there exists a *unique* $g(x)\in V$ so that the above equality holds for all $f(x)\in V$.

(4) (15%) Calculate the volume of the parallelepiped:

$$P=\{\alpha x+\beta y+\gamma z \mid \alpha,\beta,\lambda\in[0,1]\}$$

spanned by $x=(2,0,1,1,0)$, $y=(0,0,2,1,2)$, $z=(1,1,1,1,1)$ in the 5-dimensional real Euclidean space.

(5) (20%) Suppose A is a *real* 4×4 matrix satisfying $A^2+I_4=0$. Find all possible Jordan canonical forms of A. Here I_4 denote the identity matrix.

(6) (15%) Denote the standard inner product of $a,b\in\mathbb{R}^n$ by $<a,b>$ and denote $\|a\|^2=<a,a>$, $S=\{a\in\mathbb{R}^n \mid 1=\|a\|\}$. Let A be a *real* $n\times n$ matrix and consider the function $f(x)=<x,Ax>, x\in S$. Suppose $a\in S$ and there exists a positive number δ such that $f(a)\ge f(x)$ for all $x\in S$ satisfying $\|x-a\|<\delta$. Can we conclude that $f(a)\ge f(x), \forall x\in S$? Prove or disprove it.

試題隨卷繳回