1. (20 points) Suppose $A(t) = (a_{ij}(t)) : \mathbb{R} \to \mathbb{R}^{n^2}$ is a differentiable matrix-valued function and for a fixed $p > 0$, $a_{ij}(t+p) = a_{ij}(t)$ for $i, j = 1, 2, \cdots, n$. Assume that $\Phi(t)$ is a fundamental matrix of the differential equation

$$X'(t) = A(t)X(t).$$

Prove that $\Phi(t)^{-1}\Phi(t+p)$ is a constant matrix.

2. (20 points) Solve the differential equation

$$\begin{cases} x_1'(t) = x_2, \\ x_2'(t) = x_3, \\ x_3'(t) = x_4, \\ x_4'(t) = -4x_1 + 5x_3, \end{cases}$$

with the initial condition $(x_1(0), x_2(0), x_3(0), x_4(0)) = (2, 0, 2, 0)$.

3. (20 points) Solve the differential equation

$$x^{(4)}(t) + 2x''(t) + x(t) = 0,$$

with the initial condition $(x(0), x'(0), x''(0), x'''(0)) = (2, 2, -2, -4)$.

4. (20 points) Suppose $\phi(t)$ is a solution of the differential equation

$$x'(t) = -x(t) + q(t),$$

for $t \geq 0$. Assume that $\int_0^\infty |q(t)|dt < \infty$. Prove that

$$\lim_{t\to\infty} \phi(t) = 0.$$

5. (20 points) Find a nontrivial solution of the differential equation

$$tx''(t) + x'(t) + 2x(t) = 0.$$

試題隨卷繳回