NOTATIONS :

S_n, the symmetric group of $n \geq 1$ letters.
A_n, the subgroup of $\mathbb{S}_n$ consisting of even permutations.
$\mathbb{F}_p$, the finite field with p elements, p a prime.
$\mathrm{Mat}_n(k)$, the ring of $n \times n$ matrices with entries from the field k.
$k[x]$, polynomial ring in one variable x over the field k.
$\mathrm{SL}(n,k)$, the group of $n \times n$ matrices of determinant 1 with entries from field k.

(1) (a) (3%) Let $\sigma \in S_9$ be the following permutation:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 4 & 5 & 6 & 7 & 2 & 9 & 8 & 1 \end{pmatrix}.$$

Write σ as a product of disjoint cycles.

(b) (3%) Let H be the cyclic subgroup of S_9 generated by σ. Consider this group H acting on the set $\{1, 2, \ldots, 9\}$, find the orbit of the element 1 .

(c) (3%) Write σ as a product of transpositions. Is σ in the group A_9?

(d) (8%) Count the number of elements in the conjugacy class of $\sigma \in S_9$.

(2) (10%) Let $\frac{1}{1+\sqrt[5]{2}+\sqrt[5]{4}} = x + y\sqrt[5]{2} + z\sqrt[5]{4} + w\sqrt[5]{8} + u\sqrt[5]{16}$. Solve x, y, z, w, u in $\mathbb{Q}$.

(3) (15%) Let $(\mathbb{Z}/N\mathbb{Z})^\times$ be the multiplicative groups of integers modulo N which consists of congruence classes of integers a relatively prime to N. Prove that $(\mathbb{Z}/105\,\mathbb{Z})^\times$ has a subgroup which is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, i.e. direct sum of 3 copies of the cyclic group of order 2.

(4) (20%) Let k be a field.

(a) Suppose matrix $A \in \mathrm{Mat}_n(k)$ such that $AB = BA$ for all $B \in \mathrm{Mat}_n(k)$. Prove that A must be a diagonoal matrix.

(b) Let $I \neq 0$ be a two-sided ideal of the matrix ring $\mathrm{Mat}_n(k)$. Show that $I = \mathrm{Mat}_n(k)$ must hold.

(5) (20%) Let p be a prime number.

(a) Count the number of 1-dimensional subspaces inside the two-dimensional $\mathbb{F}_p$-vector space $V = \mathbb{F}_p^2$.

(b) Count the number of elements in the finite group $\mathrm{SL}(2, \mathbb{F}_p)$

(c) Verify that any Sylow p-subgroup of $\mathrm{SL}(2, \mathbb{F}_p)$ is isomorphic to the cyclic group of order p.

(d) A theorem of Sylow asserts that any two Sylow p-subgroups are conjugate to each other. Use this to prove that there is one-to-one correspondence between the set of Sylow p-subgroups for $\mathrm{SL}(2, \mathbb{F}_p)$ and the set of 1-dimensional subspaces inside $V = \mathbb{F}_p^2$.

(6) (18%) Let k be the field $\mathbb{Q}(\sqrt{-3})$.

(a) Show that the polynomial $x^3 - 2$ is irreducible in $k[x]$.

(b) Prove that there is an isomorphism from the quotient ring $k[x]/(x^3-2)$ to the field $\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$ extending the ientity automorphism on k, and sending the coset $x + (x^3 - 2)$ to $\sqrt[3]{2}\, e^{2\pi i/3}$.

(c) Show that the group of all automorphisms of $\mathbb{Q}(\sqrt{-3}, \sqrt[3]{2})$ is isomorphic to the symmetric group S_3.