1. Let $\mathcal{B} = \{e^t,\ te^t,\ t^2e^t\}$, $V = \text{Span } \mathcal{B}$, and T be a linear operator on V defined by $T(f) = f'(t)$.

   (a) (5%) Find $[T]_\mathcal{B}$, the matrix representation of T with respect to $\mathcal{B}$.

   (b) (5%) Find the eigenvalues of T and a basis for each eigenspace.

   (c) (5%) Is T invertible? If it is, find $T^{-1}(c_1e^t + c_2te^t + c_3t^2e^t)$.

2. Let $V_1 = \text{Span } \{\mathbf{v}_1,\ \mathbf{v}_2\}$ and $V_2 = \text{Span } \{\mathbf{v}_3,\ \mathbf{v}_4\}$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix},\quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},\quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix},\quad \mathbf{v}_4 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}.$$

   (a) (5%) Find an orthogonal basis for V_1.

   (b) (5%) Find a basis for Null $\left([\mathbf{v}_3\ \mathbf{v}_4]^T\right)$.

   (c) (5%) Let W be the intersection of V_1 and V_2. Find a basis for W.

3. (20%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%)*:

   (a) A set V is a vector space if V satisfies the following properties: (i) V has a zero vector; (ii) whenever $\mathbf{u}$ and $\mathbf{v}$ belong to V, then $\mathbf{u}+\mathbf{v}$ belongs to V; and (iii) whenever $\mathbf{v}$ belongs to V and c is a scalar, then $c\mathbf{v}$ belongs to V.

   (b) Let B be an $m \times m$ invertible matrix and A be an $m \times n$ matrix. Then A and BA have the same reduced row echelon form.

   (c) An $n \times n$ matrix A is diagonalizable if and only if BAB^T is diagonalizable.

   (d) Let A be $n \times n$. Then rank A = rank A^2

   (e) If B is obtained from A by applying a series of elementary row operations, then A and B have the same reduced row echelon form.

   (f) Let $\mathcal{S} = \{\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ be a linearly independent subset of $\mathcal{R}^n$ and $\mathbf{u}$ be a vector in $\mathcal{S}^\perp$. Then $\{\mathbf{u},\ \mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_k\}$ is linearly independent.

   (g) Let A be an $m \times n$ matrix. If $A\mathbf{u} = A\mathbf{v}$ implies $\mathbf{u} = \mathbf{v}$, then rank $A = n$.

   (h) If $\mathbf{v}$ is an eigenvector of A^2, then $\mathbf{v}$ is an eigenvector of A.

   (i) Let A be $n \times n$. Then $\det(2A) = 2 \det A$.

   (j) If $\mathbf{v}$ is not a linear combination of $\{\mathbf{u}_1,\ \mathbf{u}_2,\ \ldots,\ \mathbf{u}_k\}$, then rank $[\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k\ \mathbf{v}]$ = 1+ rank $[\mathbf{u}_1\ \mathbf{u}_2\ \ldots\ \mathbf{u}_k]$.

4. Toss 3 dice together (each with 6 unbiased faces: 1 to 6). Let Z be the median of the results. For example, if the results are 4, 3 and 6, then the median is 4. If the results are 4, 2, and 2, the median is 2. Please find the average of Z: E[Z] (10%)

5. X and Y are two independent random variables. The cumulative density function (C.D.F) of X is shown in Figure 1 and the probability density function (P.D.F) of Y is shown in Figure 2. Please (1) find the value of a+b+c (4%) (2) find the P.D.F of Z=X+Y (6%)

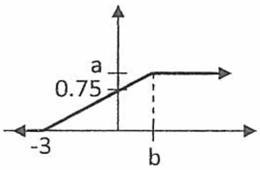

Figure 1: C.D.F. of X

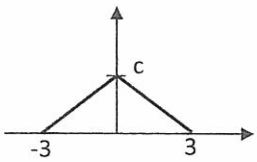

Figure 2: P.D.F. of Y

6. Let X and Y are two discrete random variables. The sample space of X is $\{x_1, x_2, \ldots, x_p\}$ and the sample space of Y is $\{y_1, y_2, \ldots, y_q\}$. The mean values of X and Y are denoted as $E[X]=M_x$ and $E[y]=M_y$, respectively. Please show that (1) if X and Y are independent of each other, then $E[XY]= M_x \cdot M_y$ (4%), (2) if $E[XY]= M_x \cdot M_y$, X and Y are NOT necessary independent of each other (6%)

7. Denote an arbitrary event $E_i \in S$, where S is the sample space of an experiment. If the number $P[E_i]$ is called as the probability of E_i, which three requirements should the function $P[\cdot]$ satisfy? 6% (Please specify each requirement in a mathematical form using only the notations given in this question)

8. Mr. Yang commutes between his home and office by foot every day. He has **ONLY** one umbrella, either placed in his home or the office. As long as the umbrella is with him and it is raining outside, he takes his umbrella when going out. If it is not raining, Mr. Yang will not take the umbrella even if his umbrella is nearby. In Taipei, it rains with a probability of 1/3 in the morning rush hour, and with a probability of 1/2 in the evening rush hour (independently). Assume that Mr. Yang has its umbrella with him at home in the morning of Day 1. Let X be the number of commutes, counted starting from Day 1, **BEFORE** Mr. Yang gets wet. That is, X=0 represents the event that Mr. Yang gets wet in the morning of Day 1, and X=1 represents the event that Mr. Yang gets wet in the evening of Day 1. (Note that Mr. Yang commutes twice per day, one from home to school in the morning, and the other from school back to home at night). Please calculate (1) Prob[Mr. Yang gets wet] 3%, (2) E[X]. (8%) (3) Is it better to have the umbrella at home in the morning of Day 1? Please justify your answer (3%)

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