Note: You can use either Chinese or English to answer the questions below. The total number of points is 100. The points for each problem are indicated in the parentheses. You need to show all the works to receive the points.

1. (20 points) The random variables X_1 and X_2 are independent of each other and both have pdf

$$f(x) = \exp(-x)I_{(0,\infty)}(x).$$

Find the pdf of X_1/X_2.

2. (30 points) The random variable X has Uniform $(-1, 1)$ distribution, and the conditional distribution of Y given $X = x$ is $N(1 + 2x, 1)$. That is,

$$f(x) = \frac{1}{2}I_{(-1,1)}(x);$$

$$f(y|x) = \frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}(y-(1+2x))^2\right\}.$$

   (a) Find Var(Y).

   (b) Find Var$(X - Y)$.

3. Suppose $X_1, \ldots, X_n$ are identically and independently distributed as

$$f(x|\theta) = 2\theta x e^{-\theta x^2}, \quad x > 0;$$

$$E(X^n) = \theta^{-n/2}\Gamma(1+\frac{n}{2}).$$

   (a) (10 points) Determine the Fisher information number for a sample of size n.

   (b) (10 points) What is the Cramér-Rao lower bound for the variance of unbiased estimator of $E(X)$?

4. Let $x_1, \ldots, x_n$ be known constants and $Y_i \sim \text{gamma}(x_i, \beta)$ be independent random variables.

   (a) (15 points) Find a uniformly most powerful size α test for $H_0 : \beta \leq \beta_0$ vs. $H_1 : \beta > \beta_0$.

   (b) (15 points) Identify the UMVUE (uniformly minimum variance unbiased estimator) for β. (Hint: start with the MLE.)