國立臺灣大學99學年度碩士班招生考試試題

(1) (13 %) Let V be the real vector space of all real-valued functions defined on $\mathbb{R}$. Let U be the subset of all even functions ($f(-x) = f(x)$ for all $x \in \mathbb{R}$), and W be the subset of all odd functions ($f(-x) = -f(x)$ for all $x \in \mathbb{R}$). Show that $V = U \oplus W$.

(2) (13 %) Let U and V be two finite dimensional vector spaces and $T : U \to V$ be a linear transformation. Let W be a subspace of V and $T^{-1}(W) = \{u \in U : T(u) \in W\}$. Show that

$$\dim T^{-1}(W) = \dim(\operatorname{im} T \cap W) + \dim \ker T.$$

(3) (20 %) A parallelepiped is bounded by the following six planes:

$$\begin{aligned} x - y - z &= 0, & x + y - z &= 0, & x - 5y + 3z &= 0, \\ x - y - z &= -4, & x + y - z &= 2, & x - 5y + 3z &= 4. \end{aligned}$$

Find the volume of this parallelepiped.

(4) (20 %) Let T be a linear transformation on the vector space $\mathbb{C}^4$. Suppose

$$T\begin{bmatrix}0\\1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\1\\1\\1\end{bmatrix}, T\begin{bmatrix}1\\0\\1\\1\end{bmatrix} = \begin{bmatrix}-2\\-6\\-6\\-4\end{bmatrix}, T\begin{bmatrix}1\\1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\-6\\-6\\-3\end{bmatrix}, T\begin{bmatrix}1\\1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\3\\3\\2\end{bmatrix}.$$

Find the Jordan canonical form of T.

(5) (20 %) Let A and B be two square matrices of degree n over a field.

(a) Prove or disprove: $\det\begin{bmatrix}A & B\\ B & A\end{bmatrix} = \det(A+B)\det(A-B)$.

(b) Prove or disprove: $\det\begin{bmatrix}A & B\\ B & A\end{bmatrix} = \det(A^2 - B^2)$.

(6) (14 %) A 2×2 integral matrix $A \in M_2(\mathbb{Z})$ is said to be of finite order if $A^k = I_2$ for some $k > 0$. Show that, if A is of finite order, then $A^k = I_2$ for some k, $0 < k \leq 6$.