國立臺灣大學99學年度碩士班招生考試試題

Problem 1. Consider the following boundary value problem of a second-order linear equation.

$$\begin{cases} y''(x) = p(x)y'(x) + q(x)y(x) + r(x), \ a \le x \le b, \\ y(a) = c_1, \\ y'(b) + ky(b) = c_2, \end{cases}$$

where a, b, c_1, c_2, and k are given real numbers and $p(x)$, $q(x)$, and $r(x)$ are given well-defined real functions. Derive a finite difference system for the problem.

(a)(5%) Discretize the domain $[a, b]$ into N equal parts. Define the grid points.

(b)(10%) Write down a second order finite difference approximation of the differential equation at the interior grid points. Define your notations clearly.

(c)(5%) Write down a first or second order finite difference approximation of the boundary condition at b.

(d)(10%) Assemble the finite difference system as the form of $\mathbf{Ay} = \mathbf{b}$ by giving the matrix $\mathbf{A}$ and the vectors $\mathbf{y}$ and $\mathbf{b}$.

Problem 2. Suppose we want to solve a linear system $\mathbf{Ay} = \mathbf{b}$, where $\mathbf{A} \in \mathbb{R}^{N\times N}$ is a tri-diagonal matrix, $\mathbf{y}, \mathbf{b} \in \mathbb{R}^N$, and $N \ge 3$ is a positive integer.

(a)(15%) Write a pseudo-code in detail to solve this linear system.

(b)(5%) Analyze the storage requirement of your code.

(c)(5%) Analyze the complexity of your code in terms of N.

Problem 3. (20%) Let $f(x) = e^x$ and $p(x) = \alpha + \beta x$, with α and β arbitrary real numbers. Determine the values of α and β such that the root mean square error $r(p, f)$ in the approximation of $f(x)$ by $p(x)$ over the interval $[-1, 1]$ is minimized. Here

$$r(p, f) = \left(\frac{1}{2}\int_{-1}^{1} [f(x) - p(x)]^2\right)^{\frac{1}{2}}.$$

Problem 4. Consider evaluating

$$f(x) = \frac{1 - \cos(x)}{x^2}$$

for a sequence of values of x approaching 0. Following table shows the results obtained by a 10-digit decimal calculator.

x	Computed $f(x)$	True $f(x)$
0.1	0.4995834700	0.4995834722
0.01	0.4999960000	0.4999958333
0.001	0.5000000000	0.4999999583
0.0001	0.5000000000	0.4999999996
0.00001	0.0000000000	0.5000000000

(a)(8%) Suppose we get $\cos(0.01) = 0.9999500004$ on the calculator. Explain why the computed $f(0.01)$ equals 0.4999960000.

(b)(7%) Give a best guess why the computed $f(0.00001)$ equals 0.0000000000.

(c)(10%) Propose a method to get better evaluations of $f(x)$ for small values of x.