請於「非選擇作答區」作答，是非與選擇題不需列推演過程，計算或證明題須列過程，請務必標示題號。

1. Let T be a linear operator on $\mathcal{P}_2$ defined by

$$T(f(x)) = f(0) + f(x) + f'(x) + f(1)x^2.$$

(a) (5%) Find $[T]_\mathcal{B}$, where $\mathcal{B}$ is the standard basis for $\mathcal{P}_2$. (b) (10%) For $f(x) = a_0 + a_1x + a_2x^2$, find $T^{-1}(f(x))$.

2. Let A be the 3×3 matrix defined below. (a) (6%) Find the eigenvalues of A, and (b) (9%) find an orthonormal basis for $\mathcal{R}^3$ consisting of eigenvectors of A.

$$A = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}$$

3. (20%) Label the following statements as being true or false. *(No explanation is needed. Each correct answer gets 2% and each wrong answer gets 0%)*:

(a) Let S be a set containing n linearly independent eigenvectors of an $n \times n$ symmetric matrix. Then S forms an orthogonal basis for $\mathcal{R}^n$.

(b) A matrix representation of a linear operator on $\mathcal{M}_{m\times n}$ is an $m \times n$ matrix.

(c) Every column of A can be *uniquely* expressed as a linear combination of the pivot columns of A.

(d) Let V be a subspace of $\mathcal{R}^n$ and W be its orthogonal complement. If $\mathbf{v}$ is a vector in V and $\mathbf{w}$ is a vector in W, then $\mathbf{v} \bullet \mathbf{w} = 0$.

(e) Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \dots\ \mathbf{a}_n]$ be a square matrix and $\mathbf{b}$ be a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. Then $\det A = \det[\mathbf{a}_1 + \mathbf{b}\ \mathbf{a}_2\ \mathbf{a}_3\ \dots\ \mathbf{a}_n]$.

(f) If V and W are subspaces of $\mathcal{R}^n$ having the same dimension, then $V = W$.

(g) Let A be an $m \times n$ matrix. Then the rank of A is n if and only if the equation $A\mathbf{x} = \mathbf{b}$ has at most one solution for each $\mathbf{b}$ in $\mathcal{R}^m$.

(h) Let A, B and C be *any* matrices such that the product ABC is defined. Then rank $(ABC) \leq$ rank B.

(i) If the reduced row echelon form of $[A\ \ b]$ contains a zero row, then $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

(j) A function from $\mathcal{R}^n$ to $\mathcal{R}^m$ is uniquely determined by its images of the standard vectors in $\mathcal{R}^n$.

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4. (5%) For the given equation $\int_0^t f(\tau)f(t-\tau) = \Lambda(t) = \begin{cases} 1-|t|, \text{for } |t| \le 1 \\ 0, \text{otherwise} \end{cases}$, please choose the **correct one answer**

(A) $f(t) = \begin{cases} 1/2, \text{for } |t| < 1/2 \\ 0, \text{otherwise} \end{cases}$; (B) $f(t) = \begin{cases} 1/2, \text{for } |t| < 1 \\ 0, \text{otherwise} \end{cases}$; (C) $f(t) = \begin{cases} 1, \text{for } |t| < 1 \\ 0, \text{otherwise} \end{cases}$;

(D) $f(t) = \begin{cases} 1, \text{for } |t| < 1/2 \\ 0, \text{otherwise} \end{cases}$; (E) None of the above.

5. (5%) You can expand the function defined by $f(x) = x^2 + 3, 0 < x < 3$ in a Fourier series, a cosine series or a sine series. Please choose the **correct answers**

(A) $f(6)=3$ for sine series; (B) $f(3)=12$ for cosine series; (C) $f(0)=3$ for Fourier series; (D) $f(-1)=4$ for Fourier series; (D) $f(-3)=12$ for cosine series.

6. (5%) Solve the initial value problem:

$$y'+y = f(t), y(0) = 5, \text{ where } f(t) = \begin{cases} 0, \text{for } 0 \le t < \pi \\ 3\cos(t), \text{ for } t \ge \pi \end{cases}.$$

Please choose the **correct answers**

(A) $y(t) = 5e^{-t}, 0 \le t < \pi$; (B) $y(t) = 5e^{-t} + \frac{3}{2}e^{-(t+\pi)}, 0 \le t < \pi$;

(C) $y(t) = 5e^{-t} + \frac{3}{2}e^{-(t+\pi)} + \frac{3}{2}\sin(t+\pi) + \frac{3}{2}\cos(t+\pi), t \ge \pi$;

(D) $y(t) = 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin(t) + \frac{3}{2}\cos(t), t \ge \pi$;

(E) $y(t) = 5e^{-t} + \frac{3}{2}e^{-(t-\pi)} + \frac{3}{2}\sin(t-\pi) + \frac{3}{2}\cos(t-\pi), t \ge \pi$

7. (15%) Please solve the differential equation $(x^2 + 2xy - y^2)dx + (y^2 + 2xy - x^2)dy = 0$.

(a) (3%) please verify that the differential equation is exact or not.

(b) (5%) please show that the integrating factor $\mu(x, y) = (x+y)^{-2}$

(c) (7%) please solve the differential equation.

8. (10%) As illustrated in the Figure 8, light rays strike a plane curve C in such a manner that all rays L parallel to the x-axis are reflected to a single point O.

(a) (5%) Assume that the angle of incidence is equal to the angle of reflection, determine a differential equation that describes the shape of the curve C.

[Hint: Please show that we can write $\phi=2\theta$ firstly.]

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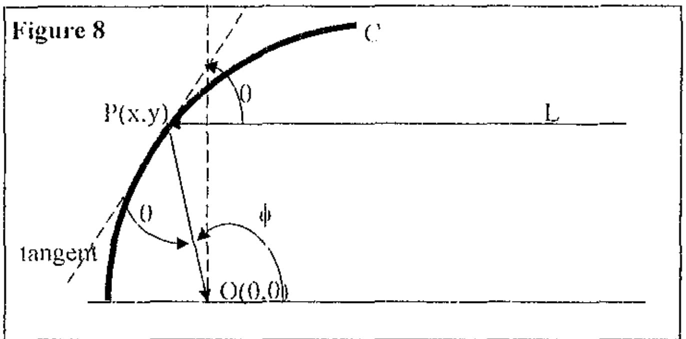
(b) (5%) Please solve the differential equation to get the function describing the curve C.

Figure 8



9. (10%) In the paraxial approximation, the light ray trajectory is almost parallel to the z-axis. The light ray equation can be expressed as

$$\frac{d}{dz}\left(n\frac{dy}{dz}\right)=\frac{dn}{dy},\ \frac{d}{dz}\left(n\frac{dx}{dz}\right)=\frac{dn}{dx}$$ where $n=n(x,y,z)$ is the refractive index.

(a) (2%) In a homogeneous medium where n is independent of x, y, z, please show that the light ray trajectory is a straight line.

(b) (8%) Let a light ray be incident into a slab graded index medium, in which $n=n_0(1-\alpha^2y^2)$ with $\alpha^2y^2<<1$, at position $y=y_0$ and with an incidence angle $\frac{dy}{dz}\cong\theta_0$. Please show that with appropriate approximation the light ray trajectory is a periodic function as in Figure 9 and find the period.

Figure 9