第 1、2 題為複選題，不倒扣但完全答對才有分數，請作答於答案卷之「選擇題作答區」，不須寫出過程，寫出亦不計分。第 3 題到第 11 題請標明題號，作答於答案卷之「非選擇題作答區」，須寫出過程及答案。禁止使用任何電子計算裝置。

1. Let $L\{\ \}$ and $L^{-1}\{\ \}$ be the Laplace and the inverse Laplace transforms, respectively. Which of the following statements are correct? (5%)
(A) If $L\{f_1(t)\} = F_1(s)$ for $s > c_1$ and $L\{f_2(t)\} = F_2(s)$ for $s > c_2$, then for $s > c_1$ and $s > c_2$, $L\{f_1(t) + f_2(t)\} = F_1(s) + F_2(s)$.
(B) If $L\{f_1(t)\} = F_1(s)$ and $L\{f_2(t)\} = F_2(s)$, then $L^{-1}\{F_1(s)F_2(s)\} = f_1(t)f_2(t)$.
(C) If $f(t)$ is piecewise continuous on $(0,\infty)$ and of exponential order, then $L\{f(t)\} = 0$ as s approaches ∞.
(D) If $f(t) = te^{-at}$, then $L\{f(t)\} = 1/(s + a)$.
(E) If $f(t) = \sinh(at)$, then $L\{f(t)\} = a/(s^2 - a^2)$.

2. Find the "parabolic" equation(s) for the following linear second-order partial differential equations. (4%)
(A) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + u = 0$, (B) $\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 9\frac{\partial^2 u}{\partial y^2} = 0$, (C) $\frac{\partial^2 u}{\partial x^2} = 9\frac{\partial^2 u}{\partial x \partial y}$,
(D) $\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial x} - 6\frac{\partial u}{\partial y} = 0$, (E) $k^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, k is a real number.

3. Determine each of the following statements is true or false. If false, give the reason.
(a) If $f(x)$ is a solution of a linear differential equation, then $\alpha f(x)$ is also a solution, where α is a constant. (5%)
(b) The set of functions, $f_1(x) = e^{x+5}$ and $f_2(x) = e^{x-2}$, is linearly independent. (5%)

4. Solve $y'' - 2y' + y = e^x$. (10%)

5. General solution of the system, $\mathbf{x}' = \begin{bmatrix} -3 & 1 \\ 2 & -4 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 3t \\ e^{-t} \end{bmatrix}$, has the form

$$\mathbf{x} = c_1\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}e^{-2t} + c_2\begin{bmatrix} k_3 \\ k_4 \end{bmatrix}e^{-5t} + \begin{bmatrix} k_5 \\ k_6 \end{bmatrix}t + \begin{bmatrix} k_7 \\ k_8 \end{bmatrix} + \begin{bmatrix} k_9 \\ k_{10} \end{bmatrix}e^{-t},$$

where c_1 and c_2 are arbitrary constants. Find the values of k_6 and k_7. (10%)

6. Expansion of $f(x) = x^2$, $0 < x < \ell$, in a Fourier series has the form $f(x) = a_1 + \sum_{n=1}^{\infty}\{a_2\cos(a_3 x) + a_4\sin(a_5 x)\}$.
(a) Find the expressions of a_1 and a_4. (8 %)
(b) Plot graphs of $f(x)$ v.s. x from -3ℓ to 3ℓ when $f(x)$ is expanded as cosine series, sine series, and Fourier series. (3%)

7. For any matrix A, let $\mathcal{N}(A)$ denote its null space. In the real space $\mathcal{R}^n$, define the inner product $\langle \mathbf{x},\mathbf{y}\rangle = x_1y_1 + \cdots + x_ny_n$ and 2-norm $\|\mathbf{x}\| = \langle \mathbf{x},\mathbf{x}\rangle^{1/2}$ for all vectors $\mathbf{x} = [\, x_1 \cdots x_n \,]^T$ and $\mathbf{y} = [\, y_1 \cdots y_n \,]^T$. Suppose S is a subspace of $\mathcal{R}^n$. Let $S^\perp$ be the orthogonal complement of S in $\mathcal{R}^n$ with respect to the inner product $\langle\cdot,\cdot\rangle$. Consider a real 4×4 matrix A with $\mathcal{N}(A)$ spanned by the set $\{[\,2\ \ 0\ \ 2\ \ -1\,]^T, [\,1\ \ 2\ \ 0\ \ -1\,]^T, [\,3\ \ -1\ \ 4\ \ -1\,]^T\}$.
(a) What is the rank of A? (5%)
(b) Find an orthonormal basis $\mathcal{B}$ for $\mathcal{N}(A)^\perp$. (5%)
(c) With the above information, judge which one of the following three conditions can uniquely determine the matrix A, and find the unique A under that condition: **Condition I:** $A = A^T$ and $\det(A) = 0$; **Condition II:** $A = A^T$ and A has an eigenvalue 1; **Condition III:** $A = -A^T$. (10%)
(d) What is the least square error solution of $A\mathbf{x} = [\,1\ \ 1\ \ 1\ \ 1\,]^T$ for the unique matrix A obtained in (c)? (5%)

8. Give an example of a linear map L from $\mathcal{R}^5$ to $\mathcal{R}^2$ whose null space equals $\{[\, x_1 \cdots x_5 \,]^T \in \mathcal{R}^5 : x_1 = 3x_2, x_3 = x_4 = x_5\}$, *or* prove that no such linear maps exist. (4%)

9. Give an example of a linear operator T from $\mathcal{R}^2$ to itself such that T has no real eigenvalues, *or* prove that no such operators exist, i.e., every linear operator $T \in \mathcal{L}(\mathcal{R}^2)$ has at least one real eigenvalue. (3%)

10. Let $\mathcal{P}_m(\mathcal{R})$ denote the inner-product space of all polynomials with real coefficients and degree at most m, with the inner product defined by $\langle p, q\rangle = \int_0^1 p(x)\,q(x)\,dx$, $\forall p, q \in \mathcal{P}_m(\mathcal{R})$. Apply the Gram-Schmidt procedure to the basis $\{1, x, x^2\}$ to produce an orthonormal basis of $\mathcal{P}_2(\mathcal{R})$. (10%)

11. Let T be a linear operator from $\mathcal{C}^n$ to itself and $\lambda \in \mathcal{C}$. Let $\mathcal{N}(T)$ denote the null space of T. Prove that for every basis of $\mathcal{C}^n$ with respect to which T has an upper-triangular matrix, λ appears on the diagonal of the matrix of T precisely $\dim\mathcal{N}((T - \lambda I)^n)$ times, *or* disprove it by giving a counter example. (8%)

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