1. The motion of a pendulum subjected to forced oscillation is governed by the differential equation

$$mL\frac{d^2\theta}{dt^2} = -mg\sin\theta + A\delta(t)\cos\theta$$

where θ is the angle of inclination of the pendulum to the downward vertical at time t, $\delta(t)$ is the delta function.

(a) (10%) Seek the solution $\theta(t)$ such that $\theta = \pi/6$ and $d\theta/dt = 0$ for $t = 0$ in the form of a series: $\theta = c_0 + c_1 t + \cdots$ through terms of degree 4.

(b) (13%) For small values of θ, write down the linearization of the equation with respect to $\theta = 0$. Find a solution of this equation by Fourier transforms.

2 Evaluate the integrals

(a) (5%) $\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$　　(b) (5%) $\int_{-\infty}^{\infty}\frac{\sin^2 x}{x^2}dx$

3 Let $\mathbf{A}$ be a square constant matrix, and $\mathbf{A}^T$ be the transpose of $\mathbf{A}$.

(a) (5%) Show that $\mathbf{A}$ and $\mathbf{A}^T$ have the same set of eigenvalues.

(b) (10%) Let $\mathbf{x}(t) = e^{\lambda t}\mathbf{c}$ and $\mathbf{y}(t) = e^{-\mu t}\mathbf{d}$ (with $\mathbf{c}$ and $\mathbf{d}$ the constant vectors) be the solutions of the following systems of ordinary differential equations,

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{A}\mathbf{x}(t), \quad \frac{d\mathbf{y}}{dt}(t) = -\mathbf{A}^T\mathbf{y}(t),$$

respectively. Show that λ, μ are eigenvalues of $\mathbf{A}$, and if $\lambda \neq \mu$, we have $\mathbf{c}^T\mathbf{d} = 0$.

(c) (6%) Use the Laplace transform to transform the following Initial Value Problem (IVP):

$$\frac{d\mathbf{x}}{dt}(t) = \mathbf{A}\mathbf{x}(t), \quad \mathbf{x}(0) = \mathbf{x}_0,$$

into an algebraic problem.

(d) (12%) Use the technique of Laplace transform to solve the following IVP:

$$\frac{d\mathbf{x}}{dt}(t) = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{x}(t), \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

4. Consider the one-dimensional wave equation $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$ in an infinite domain.

(a) (7%) With the transformation $v = x - ct$ and $z = x + ct$, show that the wave equation becomes $\frac{\partial^2 u}{\partial v \partial z} = 0$, and thus the solution can be written as $u(x,t) = \phi(x+ct) + \psi(x-ct)$, where ϕ and ψ are arbitrary functions.

(b) (10%) With initial conditions $u(x,0) = f(x)$ and $\partial u/\partial t(x,0) = g(x)$, show that the solution becomes

$$u(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds,$$ and give physical interpretation of the result.

5. (7%) A surface in space can be written as $z = f(x,y)$. A curve on this surface can be written as $x = p(t)$, $y = q(t)$, $z = (p(t), q(t))$, where t is a parameter. What are the tangent vector of the curve and the normal vector of the surface? What is the projection of the tangent vector on a direction along the vector $\hat{i} + \hat{j}$, where $\hat{i}$ and $\hat{j}$ are unit vectors along the x and y coordinates.

6. (10%) Evaluate $\oint_C (x^2 - y^2)dx + (x^2 + y^2)dy$ if C is the square with vertices A(0,0), B(1,0), D(1,1), E(0,1).

試題隨卷繳回