1. (10%) If A is an unitary matrix, prove that all the eigenvalues of the problem $A\mathbf{x} = \lambda\mathbf{x}$ have absolute value of one.

2. (10%) If the equation $ty''-(2t+1)y'+2y=0,\ t>0$ has a solution of the form e^{ct} for some c, find the general solution of this equation.

3. (10%) Find the Fourier series of the function $f(x)=\cos^3 2x+\sin^5 x$

4. (10%) Find the value of $(1+i)^{2-i}$ in the form $x+iy$

5. (20%) Given a matrix A as $A=\begin{bmatrix}1\\0\\2\end{bmatrix}\begin{bmatrix}1 & 2 & 3\end{bmatrix}$, find

   (a) the rank of A
   (b) det(A)
   (c) the eigenvalues and eigenvectors of A

6. (20%) Consider the equation of the form:
   $[r(x)y']'+[q(x)+\lambda\, p(x)]y=0,\quad a\le x\le b$ ------(1)
   where $r(x), p(x), q(x)$ are known continuous functions, λ is a parameter and $p(x)>0$. And if the boundary conditions are given by:
   $\begin{cases}k_1 y(a)+k_2 y'(a)=0\\ \ell_1 y(b)+\ell_2 y'(b)=0\end{cases}$ ---------(2)
   where k_1, k_2 (also l_1, l_2) are given constants which are not both zero. This is called the Sturm-Liouville problem. Now let y_m and y_n be eigenfunctions that correspond to distinct eigenvalues λ_m and λ_n, respectively. Prove that y_m and y_n are orthogonal on that interval with respect to the weight function $p(x)$.

7. (20%) The vibration of a semi-infinite string is governed by $\dfrac{\partial^2 u}{\partial x^2}=\dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2}$, where c is a constant and u is the displacement. The string is initially at rest on the x-axis from $x=0$ to $x=\infty$ and for time $t\ge 0$, the end of the string is excited by a force $f(t)$, i.e., $u(0,t)=f(t)$. Assume the displacement of the string at infinity is zero and then find the displacement of the string.

試題隨卷繳回