1. (15%) Consider the one dimensional heat transfer problem

$$\frac{\partial u}{\partial t}=\alpha^2\frac{\partial^2 u}{\partial x^2},\quad 0<x<10$$

with boundary conditions $u_x(0,t)=u_x(10,t)=0$

(a) What's the physical interpretation of the given boundary conditions?

(b) Find the most general solution to this boundary value problem.

(c) Given $u(x,0)=x^2$, find $u(5,t\to\infty)$.

2. (15%) Consider the one dimensional wave propagation

$$u_{xx}=u_{tt},\quad -\infty<x<\infty,\quad t>0$$

with initial conditions $u(x,0)=0$ and $u_t(x,0)=g(x)$, where $g(x)$ is a given function.

(a) Show that $u(x,t)=G(x+t)-G(x-t)$ satisfies the above wave equation and initial conditions for a suitable function $G(x)$. How are $G(x)$ and $g(x)$ related?

(b) Find $u(x,t)$ if $u_t(x,0)=g(x)=\dfrac{x}{1+x^2}$.

3. (20%) Find the solution $y(x)$ of the following initial-valued problems:

$$\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=3\exp(2x)+2x^2-7,\quad y(0)=1,\quad \frac{dy}{dx}(0)=0$$

by using (a) the method of undetermined coefficients (b) the Laplace transform. Show your derivation STEP BY STEP.

4. (20%) Answer the following questions:

(a) Let R be the real linear space consisting of all real numbers, and S be the set consisting of all positive real numbers and zero. Is S a subspace of R? If not, why not?

(b) Are the following vectors in the four dimensional real vector space, R^4, linearly dependent or independent: (1,2,−1,−1), (−2,−3,2,1), (−5,−2,5,−3)? If yes, why? If not, why not?

(c) Let A be a 4x4 matrix and $A=\begin{pmatrix}0&1&0&0\\7&0&1&0\\0&0&0&1\\-6&6&0&1\end{pmatrix}$. Find a nonsingular matrix P that can diagonalize A and show the diagonalized form of matrix A.

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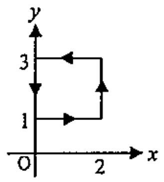
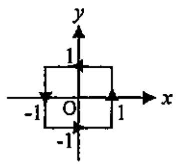
5. Write down the answers to the following questions. (Derivations are not required.)

(a) (3%) Let $\underline{\mathbf{F}} = \dfrac{x}{x^2+y^2}\,\underline{\mathbf{i}} + \dfrac{y}{x^2+y^2}\,\underline{\mathbf{j}}$ be a 2-D vector field. Evaluate $\oint_C \underline{\mathbf{F}} \cdot d\underline{\mathbf{r}}$ over a closed contour C given by:



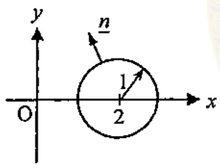
(b) (3%) Re-do the above problem, but with C given by:



(c) (3%) Find the flux of $\underline{\mathbf{v}} = \sin(2x)\,\underline{\mathbf{i}} - 4y\cos^2(x)\,\underline{\mathbf{j}} + z\underline{\mathbf{k}}$ across the surface of the sphere: $x^2+y^2+z^2=4$.

(d) (3%) Let $\underline{\mathbf{F}}$ be a conservative vector field and ϕ be its potential function, evaluate $\nabla \times (\phi \underline{\mathbf{F}})$.

(e) (3%) Let $\phi(x,y) = \ln(\sqrt{x^2+y^2})$, evaluate the line integral $\oint_C \dfrac{\partial \phi}{\partial n} d\ell$ over a closed contour C given by:



where $\underline{n}$ denotes unit vector normal to the circle C.

<hint>: Note that ϕ satisfies a Laplace equation within C.

6. Let $z = x + iy$ denote the complex variable, $\bar{z} = x - iy$ the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. (Derivations are not required.)

(a) (3%) $f(z) = z\bar{z}$ is an analytic function in the region $|z| < \infty$. (True or False)

(b) (3%) Find the residue of the complex function $f(z) = e^{1/z}/(z^2+1)$ at $z = 1$.

(c) (3%) Write down the Taylor series expansion of $f(z) = 1/(z^2+4)$ about $z = 0$.

(d) (3%) Evaluate the complex integral $\oint_C \dfrac{\bar{z}}{z(z-2i)} dz$ over C: $|z| = 1$.

(e) (3%) Evaluate the real integral $\displaystyle\int_{x=-\infty}^{x=+\infty} \frac{dx}{(x-1)(x^2+1)}$.

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