※ 注意：請於試卷上依序作答，並應註明作答之部份及其題號。

Problem 1. Matrix problem [25 %]

Given the square matrix $\mathbf{A}=\begin{pmatrix}-5 & 9\\ -6 & 10\end{pmatrix}$

a) Find the eigenvalues of **A**

b) Find the eigenvectors of **A**

c) Diagonalize matrix **A**, i.e. find matrices **P** and **D** such that $\mathbf{A}=\mathbf{PDP}^{-1}$, and find $\mathbf{A}^{30}$

d) Find $e^{\mathbf{A}}$

e) Find cos**A**

Problem 2. Gauss theorem and Green formulas [25 %]

Consider a volume V, its bounding surface S, and associated unit outward normal vector **n**, and let **F** be a differentiable vector field.

a) Find the integrand I such that $\iiint_V I\,dV=\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$

b) Use your result from a) to derive Green's second formula

$$\iiint_V(\phi\Delta\psi-\psi\Delta\phi)\,dV=\iint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dS$$

where ψ and ϕ are twice differentiable functions, and $\Delta=\vec{\nabla}\cdot\vec{\nabla}=\nabla^2$

c) Let $\phi=\frac{1}{4\pi}\frac{1}{r}$, $r=\sqrt{x^2+y^2+z^2}$, such that $\Delta\phi=-\delta$, where $\delta=$ Dirac delta function, and let ψ be such that $\Delta\psi=0$. Use your result from b) to derive Green's third formula

$$\psi(x,y,z)=\iint_S\left[\frac{1}{4\pi}\frac{1}{r}\frac{\partial\psi}{\partial n}-\frac{1}{4\pi}\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\psi\right]dS$$

Problem 3. Initial value problem [25 %]

Consider the linear system of ordinary differential equations

$$\frac{d}{dt}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}=\mathbf{A}\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$$

subject to initial conditions $x(0)=1$, $y(0)=-1$.

Find $x(t), y(t)$, $t\geq 0$, when

a) $\mathbf{A}=\begin{pmatrix}-2 & 0\\ 0 & 1\end{pmatrix}$	b) $\mathbf{A}=\begin{pmatrix}-1 & 3\\ 0 & 2\end{pmatrix}$
c) $\mathbf{A}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$	d) $\mathbf{A}=\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}$

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Problem 4. Boundary value problem [25 %]

Groundwater flow in a horizontal leaky aquifer is governed by two simultaneous first order ordinary differential equations (ODEs) forming the system:

$$\begin{cases} -K\dfrac{dh}{dx}=q \\ \dfrac{dq}{dx}=k(H-h) \end{cases} \qquad (*)$$

where $h(x)=$ groundwater pressure head, $q(x)=$ horizontal groundwater flow rate, and where k, K, and H are constants taking the following values:

$$k=1, \quad K=100, \quad H=2.$$

a) Transform the system (*) into a single second order ODE for unknown $h(x)$.

b) Find the general solution, i.e. find the most general functions $h(x)$, $q(x)$ which satisfy system (*).

c) Solve system (*) on domain $0<x<5$ under the external boundary conditions

$$h(0)=3, \quad h(5)=1$$

d) Solve system (*) on domain $-\infty<x<0 \cup 0<x<\infty$ under the internal and external boundary conditions

$$h(0)=1, \qquad \lim_{x\to+\infty} h(x)=\lim_{x\to-\infty} h(x)=H=2,$$

e) Draw (plot) your solutions to questions c) and d).

試題隨卷繳回