1. (20 points) Find the general solution of the system

$$x'(t) = x + 2y - \sin t,$$
$$y'(t) = x + \sin t.$$

2. (20 points) Let (x(t), y(t), z(t)) be a solution of the system

$$x'(t) = -2x - z^2 - y^2,$$
$$y'(t) = -5y - x^2,$$
$$z'(t) = -2z - 2y^2,$$

such that $x^2(0)+y^2(0)+z^2(0) \leq 2$. Show that $(x(t), y(t), z(t)) \to (0,0,0)$ as $t \to 0$.

3. (20 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(0) = 0$ and $f'(0) > 0$. Consider the initial value problem

$$\frac{dx(t)}{dt} = f(x),$$
$$x(0) = 2^{-2009}.$$

Show that there exists some $\delta > 0$, depending on $f(x)$, such that $x(t) \geq \delta$ for all $t > 0$.

4. (20 points) Is it possible to find two continuous functions, $p(x)$ and $q(x)$, such that $y = x^3$ is a solution of $y'' + p(x)y' + q(x)y = 0$ in $\mathbb{R}$? If this is possible, find such $p(x)$ and $q(x)$. If this is impossible, show your proof.

5. Assume that $y_1(x)$, $y_2(x)$ and $y_3(x)$ are three solutions of the second order linear differential equation

$$\frac{d^3y}{dx^3} + a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x) = 0, \qquad \text{for } a \leq x \leq b.$$

We define a function of x called the Wronskian W as

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ \frac{dy_1}{dx} & \frac{dy_2}{dx} & \frac{dy_3}{dx} \\ \frac{d^2y_1}{dx^2} & \frac{d^2y_2}{dx^2} & \frac{d^2y_3}{dx^2} \end{vmatrix}.$$

(a)(10 points) Derive a differential equations for the Wronskian W.
(b)(10 points) Show that the value of the Wronskian $W(y_1, y_2, y_3)$ is either identically equal to zero or is never equal to zero for all x on $a \leq x \leq b$.

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