There are problems A to F with a total of 100 points. Please write down your computational or proof steps clearly on the answer sheets.

A. (14 points) Evaluate the triple integral $\int_B e^{|x-a|}dx$, where $B = \{x \in \mathbf{R}^3 \mid |x| \le 1\}$, and $a \in \mathbf{R}^3$ is a given point on the unit sphere, i.e. $|a| = 1$.

B. (14 points) Prove that the 1-form $\omega = \frac{2(x^2-y^2-1)dy - 4xydx}{(x^2+y^2-1)^2+4y^2}$ is a closed form. Then evaluate the line integral $\oint_\gamma \omega$, where γ is the cardiod curve in the plane defined by the polar equation $r = 1+\cos\theta$ oriented by decreasing θ.

C. (14 points) Let α be a positive constant. Determine whether the intgral $\int_0^\infty \frac{(\sin x^2)(1-\cos x^2)}{x^\alpha}dx$ converges absolutely, or converges conditionally, or diverges.

D. (14 points) Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuously differentiable. Assume that f satisfies

$$|f(x) - f(y)| \ge C|x-y| \text{ for all } x, y \in \mathbf{R}^n, \text{ where } C > 0 \text{ is some constant .}$$

Prove that $f^{-1} : \mathbf{R}^n \to \mathbf{R}^n$ exists, and it is continuous. Must f^{-1} be differentiable?

E. (14 points) Assume that $f_n : \mathbf{R} \to \mathbf{R} (n = 1, 2, 3, \cdots)$ is a sequence of differentiable functions such that each $f_n(x)$ is a solution of the equation $y'(1+x^2+xy+y^2) = 1$. If $\sup_n |f_n(\frac{1}{n})| < \infty$, prove that there exists a subsequence $f_{n_k}(x)$ such that $\lim_{k\to\infty} f_{n_k}(x) = f(x)$ exists for $x \in \mathbf{R}$. This limit $f(x)$ must also be a solution of $y'(1+x^2+xy+y^2) = 1$.

F. Determine which of the following statements is true. Prove your assertion. Each has 6 points.

(a) For a subset $A \subset \mathbf{R}^n$, ∂A denotes its boundary. When A is bounded, $\partial(\partial A) = \partial A$ must be true.

(b) Given a sequence $a_n \ge 0 (n = 0, 1, 2, \cdots)$. Then $\lim_{x\to 1-} \sum_{n=0}^\infty a_n x^n = \sum_{n=0}^\infty a_n$.

(c) There exists *no* function $u(x, y)$ which is C^2 in $\mathbf{R}^2$, and satisfies

$$u(x,y) = 0 \text{ on } x^2+xy+y^2 = 1,\ u(x,y) \ge 0 \text{ and } u_{xx}+u_{yy} = 1+u^2 \text{ for } x^2+xy+y^2 < 1.$$

(d) Let all the partial derivatives of $z = f(u, v)$ exist. Suppose that $u = \phi(x, y)$ and $v = \psi(x, y)$ are differentiable. Then all partial derivatives of $z = f(\phi(x, y),\ \psi(x, y))$ exist, and $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x}$.

(e) Let $f(x, y)$ be a bounded function defined on the rectangle $K = [0,\ 1] \times [0,\ 1]$. Then f is Riemann integrable in K iff both the iterated integrals

$$\int_0^1 \left(\int_0^1 f(x,y)dy\right)dx \quad \text{and} \quad \int_0^1 \left(\int_0^1 f(x,y)dx\right)dy$$

exist, and equal.

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