※ 注意：請於試卷上「非選擇題作答區」標明大題及小題題號，並依序作答。

Notation: We denote by $\mathbf{R}$ the set of real numbers. For any positive integer $n$, $I_n$ is the identity matrix and $0_n$ is the zero matrix in $\mathrm{M}_n(\mathbf{R})$.

**Problem 1** (15pts). Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation defined by $T(v) = A \cdot v$, where

$$A = \begin{pmatrix} 3 & -5 & 1 & 1 \\ 1 & 5 & -1 & 3 \\ 2 & 0 & 0 & 2 \end{pmatrix} \in \mathrm{M}_{3\times 4}(\mathbf{R}).$$

(1) (10pts) Find bases of $\operatorname{Ker} T$ (the kernel of $T$) and $\operatorname{Im} T$ (the image of $T$).

(2) (5pts) Verify if the vector $\begin{pmatrix} -5 \\ 5 \\ 0 \end{pmatrix}$ belongs to $\operatorname{Im} T$.

**Problem 2** (25pts). Let

$$A = \begin{pmatrix} 0 & 1 & -1 \\ 3 & -4 & 1 \\ 3 & -8 & 5 \end{pmatrix}.$$

(1) (15pts) Find an invertible matrix $Q \in M_3(\mathbf{R})$ such that

$$Q^{-1}AQ = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 12 \\ 0 & 1 & 1 \end{pmatrix}.$$

(2) (10pts) Find an invertible matrix $P \in M_3(\mathbf{R})$ such that $P^{-1}AP$ is a diagonal matrix.

**Problem 3** (10pts). Let $A, B \in \mathrm{M}_3(\mathbf{R})$ such that

$$\det(A) = \det(A + B) = \det(A - B) = 0.$$

Show that $\det(A + 2B) = 3\det(B)$.

**Problem 4** (15pts). Let $A \in \mathrm{M}_n(\mathbf{R})$. If $\operatorname{rank} A + \operatorname{rank}(A - I_n) = n$, show that $\operatorname{Tr}(A) = \operatorname{rank} A$.

**Problem 5** (15pts). Let $A = (a_{ij}) \in \mathrm{M}_n(\mathbf{R})$. For each positive integer $k$, we define $A^{[k]} := (a_{ij}^k)$. For example, if $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, then $A^{[2]} = \begin{pmatrix} 1 & 4 \\ 9 & 16 \end{pmatrix}$.

Show that if $A^k = A^{[k]}$ for $k = 1, 2, \ldots, n+1$, then $A^k = A^{[k]}$ for all $k$.

**Problem 6** (20pts). Let $A, B \in \mathrm{M}_n(\mathbf{R})$ such that

$$AB - BA = \alpha A$$

for some non-zero real number $\alpha \neq 0$.

(1) (5pts) Show that $A^kB - BA^k = \alpha k \cdot A^k$ for all positive integer $k$.

(2) (15pts) Prove that $A^k = 0_n$ for some positive integer $k$.

試題隨卷繳回