

※ 注意：請於試卷上「非選擇題作答區」標明大題及小題題號，並依序作答。

1. (10 %) Find all points (a, b, c) in \mathbb{R}^3 for which the system

$$\begin{cases} 2x + 4y + 6z = a \\ 4x + 5y + 6z = b \\ 7x + 8y + 9z = c \end{cases}$$

has at least one solution.

2. (15%) Let T be a linear operator on \mathbb{R}^3 and $\beta = \left\{ \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Assume that $[T]_{\beta} = \begin{pmatrix} 0 & 2 & 8 \\ -1 & 4 & 6 \\ 0 & -1 & -1 \end{pmatrix}$. Determine the adjoint operator of T with respect to the standard basis of \mathbb{R}^3 .

3. (15 %) Let $A := \begin{pmatrix} 0 & 2 & 3 & 4 & 5 & 6 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & 0 & 0 \\ 6 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ and I be the 6×6 identity matrix.

- (a) Find the eigenvalues of A .
(b) Compute the determinant of $I + A$.
(Justify your answers!)

4. (20 %) Let $A \in M_{n \times n}(F)$ where F is a field.

- (a) Show that if k is the largest integer such that some $k \times k$ submatrix of A has a nonzero determinant, then $\text{rank}(A) = k$.
(b) If A is nilpotent of index m (that is, $A^m = 0$ but $A^{m-1} \neq 0$), and if, for each vector v in F^n , W_v is defined to be the subspace generated by $v, Av, \dots, A^{m-1}v$, how large can the dimension of W_v be? (Justify your answer.)

5. (20%) Suppose that S, T are subspaces of a finite-dimensional vector space V .

- (a) Show that $\dim S + \dim T = \dim(S \cap T) + \dim(S + T)$.
(b) Let $P_3(\mathbb{R})$ be the space of all polynomials over \mathbb{R} of degree less than or equal to 3. Let S be the subspace of $P_3(\mathbb{R})$ spanned by $1 - x + x^2$, $x - x^2 + x^3$ and let T be the subspace spanned by $1 + x$, $x + x^2$, $x^2 + x^3$. Find bases for $S + T$ and $S \cap T$.

6. (20 %)

- (a) Find the Jordan canonical form of $A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}$.

- (b) How many possible Jordan forms are there for a 7×7 complex matrix with characteristic polynomial $(x + 1)(x - 2)^4(x + 3)^2$?