- Any device with a computer algebra system is prohibited during the exam.
- There are FOUR questions in total. Label the question numbers **clearly** on your work.
- Answer all questions. You will have to **show all of your calculations or reasoning** to obtain credits.

1. For a point P on a smooth plane curve $\mathcal{C}$, the *osculating circle* $\mathcal{O}$ to $\mathcal{C}$ at P is defined to be the circle that satisfies two conditions :

   (1) the circle $\mathcal{O}$ and the curve $\mathcal{C}$ share the same tangent line at P;

   (2) the rate of change of the slope of tangent of $\mathcal{C}$ at P equals that of $\mathcal{O}$ at P.

   Now consider the curve $\mathcal{C}$: $y = 1 - x + \tan(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

   (a) (5%) Find $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}}$ and $\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{4}}$.

   (b) (10%) Find the center and the radius of the osculating circle $\mathcal{O}$ to $\mathcal{C}$ at the point whose x-coordinate equals $\frac{\pi}{4}$.

2. Consider $f(x) = \int_1^x \frac{1}{\sqrt{1+t^5}}\,dt$ for $x \geq 1$.

   (a) (5%) Prove that $f(x) < f(y)$ whenever $1 \leq x < y$.

   (b) (10%) Prove that $f(x) < \frac{2}{3}$ for all $x \geq 1$ and also show that $f(4) > \frac{1}{3}$.

   (c) (10%) Let $g(u)$, where $0 < u < f(4)$, be a function such that $f(g(u)) = u$. Find $g'(u)$ and $g''(u)$ in terms of $g(u)$.

   (d) (10%) Let $h(u) = e^u - g(u)$, where $0 < u < f(4)$. Prove that $h(u)$ does not have a local minimum value.

3. Let $a \in \mathbb{R}$ and $f(x,y,z)$, $g(x,y,z)$ be two smooth functions $\mathbb{R}^3 \to \mathbb{R}$. Consider the optimization problem :

   $$\text{Maximize } f(x,y,z) \text{ subject to } g(x,y,z) = a.$$

   Suppose, for each $a \in \mathbb{R}$, it is known that

   (1) the maximum value $f_{\max}(a)$ of $f(x,y,z)$ is attained at $\mathbf{r}(a) = (x^*(a), y^*(a), z^*(a))$. i.e. $f_{\max}(a) = f(\mathbf{r}(a))$;

   (2) there exists $\lambda(a) \in \mathbb{R}$ such that $\nabla f(\mathbf{r}(a)) = \lambda(a) \cdot \nabla g(\mathbf{r}(a))$.

   Answer the following questions.

   (a) (10%) Prove that $\frac{d f_{\max}}{da} = \lambda(a)$.

   (b) (10%) It is known that a differentiable function $f(x,y,z)$, when restricted to the surface $z = x^3 + y^4 - 3xy^2 + 1$, attains a global maximum value at $(-1,1,4)$. Moreover, $f_y(-1,1,4) = 20$. Use linearization to estimate the change of the maximum value when $f(x,y,z)$ is restricted to the surface $z = x^3 + y^4 - 3xy^2 + 0.8$ instead.

4. Evaluate the following integrals.

   (a) (5%) $\int_0^1 \frac{1}{\sqrt{x(1-x)}}\,\mathrm{d}x$.

   (b) (5%) $\int_1^2 \frac{1}{x^5} \cdot e^{-1/x^2}\,\mathrm{d}x$.

   (c) (10%) $\iint_R \sqrt{4 - x^2 - y^2}\,\mathrm{d}A$ where R is the region enclosed by $x^2 + y^2 = 2x$.

   (d) (10%) $\int_0^1 \int_{\sqrt{x}}^1 \int_0^{1-y} \sin((z-1)^4)\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x$.

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