國立臺灣大學 114 學年度碩士班招生考試試題

- Any device with a computer algebra system is prohibited during the exam.
- There are FIVE questions in total. Label the question numbers **clearly** on your work.
- Answer all questions. You will have to **show all of your calculations or reasoning** to obtain credits.

1. Let $f(x,y) = x^3 + y^4 - 3xy^2 + 1$.
   (a) (10%) Find all critical points of $f(x,y)$ and classify them as local maximum, minimum or saddle point.
   (b) (10%) It is known that a differentiable function $g(x,y,z)$, when restricted to the surface $z = f(x,y)$, attains a global maximum value at $(-1,1,4)$. Moreover, $g_y(-1,1,4) = 20$. Find $\nabla g(-1,1,4)$ and hence estimate the value of $g(-1,1.1,3.9) - g(-1,1,4)$.

2. (15%) Let $f(x,y)$, $g(u,v)$ be two differentiable functions with continuous second order partial derivatives such that

$$f(x,y) = g(x^2 - y^2, 2xy).$$

   Suppose $\dfrac{\partial^2 f}{\partial x^2} + \dfrac{\partial^2 f}{\partial y^2} = 0$, find $\dfrac{\partial^2 g}{\partial u^2} + \dfrac{\partial^2 g}{\partial v^2}$.

3. (15%) Heron's formula states that the area of a triangle with sides of length $a, b, c > 0$ is given by

$$S = \sqrt{s(s-a)(s-b)(s-c)} \text{ where } s = \frac{a+b+c}{2}.$$

   Use the method of Lagrange's multipliers to find the maximum possible area of a right-angled triangle whose perimeter equals a fixed constant P. Express your answers in P.

4. (20%) Express the iterated integral

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^3 \int_0^{\sqrt{16-r^2}} (1+z^2) r^4 \sin^2\theta \cos\theta \, dz \, dr \, d\theta$$

   as an iterated integral
   (a) (10%) in spherical coordinates,
   (b) (10%) in $dx\,dz\,dy$.
   Do not evaluate the integral.

5. (30%) For each of the following statements, determine whether it is true or false.
   - If it is true, prove it.
   - If it is false, justify by giving a concrete counter-example.

   (a) If a smooth $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\text{curl}(\mathbf{F}) = \mathbf{0}$ and $\text{div}(\mathbf{F}) = 0$, then $\mathbf{F}$ is a constant vector field (which means each of its components is a constant function).
   (b) Suppose $\mathbf{F}$ is a continuously differentiable vector field defined on $\mathbb{R}^3$ such that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed curve C in $\mathbb{R}^3$. Then we have $\iint_S \mathbf{F} \cdot d\mathbf{S} = 0$ for every closed surface S in $\mathbb{R}^3$.
   (c) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a continuously differentiable scalar function. The flux of the vector field $\mathbf{F} = \nabla f$ is non-negative across S where S is part of the level surface $f(x,y,z) = c$, oriented in the direction where f increases.
   (d) Let S be a surface which is contained in the plane $x + 2y + 2z = 0$, oriented downward. Suppose S has an area 691. Consider the field $\mathbf{F} = \langle 1, x, -x \rangle$. The flux of $\mathbf{F}$ across S equals $-\dfrac{691}{3}$.

試題隨卷繳回