1. (25 pts) Solve the following equations.
   (a) $\dfrac{dy}{dx} = 1 + (3 - x + y)^3$.
   (b) $e^x y^2 \dfrac{dy}{dx} + (1 + y^2) = 0$.
   (c) $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = 0, y(0) = 3, y'(0) = 0$.

2. (30 pts)
   Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ -4 & 5 \end{pmatrix}$.
   (a) Find $e^{\mathbf{A}t}$.
   (b) Solve the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t),\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.
   (c) Solve the system $\mathbf{x}'(t) = \mathbf{A}\mathbf{x}(t) + \begin{pmatrix} -2 \\ 1 \end{pmatrix},\ \mathbf{x}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

3. (25 pts) Assume that both $y_1(x)$ and $y_2(x)$ have continuous second derivatives on $[0, \pi]$ and satisfies

$$\text{(E1)} \quad \begin{cases} y'' + y = 0 \text{ for } x \in [0, \pi/2] \\ y' + xy = 0 \text{ for } x \in [\pi/2, \pi]. \end{cases}$$

   (a) Find the solution $y_1(x)$ with $y_1(\pi) = 2$.
   (b) Is it possible that $y_2(0) = 1$?
   (c) Let $S = \{y : [0, \pi] \to \mathbb{R} \,|\, y \text{ has a continuous second derivative and satisfies (E1)}\}$. Show that S is a vector space over $\mathbb{R}$ and find its dimension.

4. (20 pts) Let y satisfy

$$y'' + y^3 = 0.$$

   (a) Show that

$$(y')^2 + \frac{1}{2}y^4 = \text{constant}.$$

   (b) Assume $y(0) \neq 0$. Show that y is a periodic function.

試題隨卷繳回