※ 注意：請於試卷上「非選擇題作答區」內依序作答，並應註明作答之大題及其題號。

Instructions.

- There are two problems in two pages.
- In a problem, if an exercise depends on the conclusions of other exercises that precede it, you may assume these conclusions without solving them.

Problem 1 (80 points). Let m and n be two positive integers. The $\mathbf{C}$-vector space of matrices of size $m \times n$ with coefficients in $\mathbf{C}$ is denoted by $M_{m,n}(\mathbf{C})$. We also set $M_n(\mathbf{C}) = M_{n,n}(\mathbf{C})$.

The aim of this problem is to prove the following statement.

Theorem. *Let m, n and r be positive integers with $r \le m \le n$. Let $V \subset M_{m,n}(\mathbf{C})$ be a $\mathbf{C}$-linear subspace. Assume that every matrix A in V satisfies* $\operatorname{rank} A \le r$. *Then*

$$\dim V \le nr.$$

(1) Show that it suffices to prove the theorem for $m = n$.

(2) Assume that $m = n$. Show that we can assume that V contains the block matrix

$$R = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}$$

where I_r is the identity matrix of rank r.

From now on, we assume that $m = n$, and that $R \in V$.

(3) Let

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \in V$$

be a block matrix in V with $M_{11} \in M_r(\mathbf{C})$. Show that

$$M_{22} = 0 \quad \text{and} \quad M_{21}M_{12} = 0.$$

(Hint: you may consider the $(r+1) \times (r+1)$ minors of $M + tR$ for $t \in \mathbf{C}$.)

(4) Let

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & 0 \end{pmatrix} \in V, \quad B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & 0 \end{pmatrix} \in V$$

be two block matrices with $A_{11}, B_{11} \in M_r(\mathbf{C})$. Show that

$$A_{21}B_{12} + B_{21}A_{12} = 0.$$

(5) Let $\phi : V \to M_{r,n}(\mathbf{C})$ be the map sending a matrix $M \in V$ to its first r rows. Define the $\mathbf{C}$-linear subspace

$$W = \left\{ \begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \in V \;\middle|\; A_{21} \in M_{n-r,r}(\mathbf{C}) \right\} \subset V,$$

and let $s = \dim W$. Show that

$$\dim \phi(V) \leq nr - s,$$

by considering the map

$$\psi : W \to M_{r,n}(\mathbf{C})^{\vee}$$

$$\begin{pmatrix} 0 & 0 \\ A_{21} & 0 \end{pmatrix} \mapsto T_{A_{21}}$$

to the dual of $M_{r,n}(\mathbf{C})$, where $T_{A_{21}}$ is the linear form defined by

$$T_{A_{21}}(B_{11}, B_{12}) = \operatorname{Tr}(A_{21}B_{12})$$

for every block matrix $(B_{11}, B_{12}) \in M_{r,n}(\mathbf{C})$ with $B_{11} \in M_r(\mathbf{C})$.

(6) Conclude that

$$\dim V \leq nr.$$

(7) Show that the inequality in the theorem is optimal. More precisely, for all positive integers m, n and r with $r \leq m \leq n$, construct $V \subset M_{m,n}(\mathbf{C})$ as in the theorem such that

$$\dim V = nr.$$

Problem 2 (20 points). Let V be a nonzero vector space over a field F. Let

$$B : V \times V \to F$$

be a non-degenerate symmetric bilinear form on V, and let

$$q : V \to F$$

$$v \mapsto B(v, v)$$

be the associated quadratic form. For every $x \in F$, we say that q represents x if $q(v) = x$ for some nonzero $v \in V$.

(1) Suppose that q represents 0. Show that q represents every element of F. (Hint: Consider $q(cv + w)$ with $c \in F$ and some suitable $w \in V$.)

(2) Show that B extends to a non-degenerate symmetric bilinear form on $V \oplus F$ whose associated quadratic form represents every element of F.