1. (20%) Let $\gamma(s) : I \to \mathbb{R}^3$ be a curve parametrized by arc-length, where I is some open interval in $\mathbb{R}^1$. Suppose that γ has positive curvature for all $s \in I$. Denote by $\mathbf{T}(s), \mathbf{N}(s)$ and $\mathbf{B}(s)$ its Frenet frame. We call the line passing through $\gamma(s)$ with direction $\mathbf{B}(s)$ the *binormal line* of γ at s.

   If $\gamma(I)$ lies in a sphere, and all its binormal lines are tangent to this sphere, show that γ is contained in a great circle of this sphere.

2. (20%) Let $\Sigma \subset \mathbb{R}^3$ be a connected, regular surface. Suppose that all the geodesics of Σ are plane curves. Prove that Σ is contained either in a plane or a sphere.

3. (20%) Does there exist a regular surface $F(u, v) : (-\frac{\pi}{4}, \frac{\pi}{4}) \times (0, 1) \to \mathbb{R}^3$ with first fundamental form

$$(\mathrm{d}u)^2 + (\cos u)^2 (\mathrm{d}v)^2$$

   and second fundamental form

$$(\cos u)^2 (\mathrm{d}u)^2 + (\mathrm{d}v)^2 \text{ ?}$$

   Justify your answer.

4. (20% = 10% + 10%) Let α be a positive constant. Consider the cone

$$C_\alpha = \{(x, y, \alpha\sqrt{x^2 + y^2}) \in \mathbb{R}^3 : 0 \le x^2 + y^2 \le 1\}.$$

   (a) Compute the Gaussian curvature of C_α.

   (b) Compute the geodesic curvature of $\{(\cos\theta, \sin\theta, \alpha) : 0 \le \theta \le 2\pi\}$ in C_α.

5. (20%) Suppose that $\Sigma \subset \mathbb{R}^3$ is a closed[1] regular surface whose Gaussian curvature is positive everywhere. Suppose that γ is a simple closed geodesic on Σ, and $\Sigma \setminus \gamma$ has two connected components, A and B.

   Let $N : \Sigma \to S^2$ be the Gauss map. Show that $N(A)$ and $N(B)$ have the same area.

[1] compact without boundary