題號： 40　　　國立臺灣大學 113 學年度碩士班招生考試試題

科目： 代數

節次： 2

1. Let G be a group of order 231.

(a) **(5 points.)** Prove that a Sylow 7-subgroup of G is normal in G.

(b) **(15 points.)** Prove that the center $Z(G)$ contains a Sylow 11-subgroup of G.

2. **(15 points.)** Classify the quotient group $(\mathbb{Z}\times\mathbb{Z}\times\mathbb{Z})/\langle(4,4,8)\rangle$ according to the fundamental theorem for finitely generated abelian groups. Here $\langle(4,4,8)\rangle$ denotes the cyclic subgroup generated by $(4,4,8)$.

3. Let R be an integral domain such that every ideal is finitely generated.

(a) **(10 points.)** Prove that R satisfies the ascending chain condition on ideals. That is, prove that if $I_1 \subseteq I_2 \subseteq \dots$ is an ascending chain of ideals of R, then there exists an integer N such that $I_n = I_N$ for all $n \geq N$.

(b) **(15 points.)** Prove that every nonzero, nonunit element of R can be factored into a product of irreducibles.

4. Let $R = \mathbb{Z}[i]$. (Here $i = \sqrt{-1}$.)

(a) **(5 points.)** Prove that $4+i$ is an irreducible in R.

(b) **(5 points.)** What is the characteristic of the field $R/\langle 4+i\rangle$, and how many elements are there in $R/\langle 4+i\rangle$?

(c) **(5 points.)** Find the order of $(1+i)+\langle 4+i\rangle$ in the multiplicative group of nonzero elements in $R/\langle 4+i\rangle$.

(d) **(5 points.)** Express the multiplicative inverse of $(1+2i)+\langle 4+i\rangle$ in the form $(a+bi)+\langle 4+i\rangle$, $a, b\in\mathbb{Z}$.

5. **(20 points.)** Let p be a prime. Prove that for any nonzero element a in $\mathbb{F}_p$, the polynomial $x^p - x + a \in \mathbb{F}_p[x]$ is irreducible and separable over $\mathbb{F}_p$. (Here $\mathbb{F}_p$ denotes a finite field of p elements. *Hint*: Consider the action of the Frobenius automorphism on the roots.)

試題隨卷繳回