1. (13 points) Suppose that $Y_1, \ldots, Y_n$ are independent binomial $(m_i, p)$, where the $m_i \geq 1$ are known constants. Let

$$T_1 = \frac{\sum_{i=1}^n Y_i}{\sum_{i=1}^n m_i} \text{ and } T_2 = \frac{1}{n}\sum_{i=1}^n \frac{Y_i}{m_i}$$

be estimators of $p$.

(1a) Find the mean squared error (MSE) of $T_1$, MSE($T_1$). (5 points)

(1b) Find the mean squared error (MSE) of $T_2$, MSE($T_2$). (5 points)

(1c) Which estimator is better? (3 points)
**Hint:** the arithmetic-geometric-harmonic mean inequality,

$$\frac{1}{n}\sum_{i=1}^n x_i \geq \frac{n}{\sum_{i=1}^n \frac{1}{x_i}}$$

2. (28 points) Let $X_1, \ldots, X_n$ be independent identically distributed random variables with pdf

$$f(x) = \frac{\theta}{2(1+|x|)^{(\theta+1)}}, \ -\infty < x < \infty;$$

where the unknown parameter $\theta > 0$.

(2a) Prove that $T = \sum_{i=1}^n \log(1+|X_i|)$ is a complete sufficient statistic. (5 points)

(2b) Find the Cramer Rao lower bound (CRLB) for estimating $1/\theta$. (10 points)

(2c) Find the uniformly minimum variance unbiased estimator (UMVUE) for $\theta$. (10 points)
**Hint:** Find the distribution of $W = \log(1+|X|)$.

(2d) Find the maximum likelihood estimator (MLE) of $\theta$. (3 points)

3. (9 points) Suppose that $Y_1, \ldots, Y_n$ are iid sample from a normal distribution, $N(0, \sigma^2)$, where the population variance $\sigma^2 > 0$ is unknown. Let

$$Q = \left(\frac{\overline{Y}}{\sigma/\sqrt{n}}\right)^2 + \frac{(n-1)S^2}{\sigma^2},$$

where $\overline{Y}$ and $S^2$ are the sample mean and sample variance, respectively. Show that the distribution of $Q$ does not depend on $\sigma^2$ and use it to derive a $100(1-\alpha)\%$ confidence interval for $\sigma^2$.

4. (10 points) Let $X$ denote a random sample from a population $\pi$. Suppose that $T(X)$ is a UMP test of size $\alpha \in (0,1)$ for $H_0 : \pi = \pi_0$ and $H_1 : \pi = \pi_1$, where $\pi_0$ and $\pi_1$ are two known populations. The power of $T(X)$ when $H_1$ is true is denoted by $\beta$, where $\beta < 1$. Find a UMP test of size $1-\beta$ for $H_0 : \pi = \pi_1$ and $H_1 : \pi = \pi_0$.

5. (20 points) Let $X_1, X_2, \ldots, X_n$ and $Y_1, Y_2, \ldots, Y_m$ denote independent random variables from $N(\theta_1, \theta_3)$ and $N(\theta_2, \theta_4)$, where $\theta_1, \theta_2, \theta_3, \theta_4$ are unknown parameters.

(5a) Find an LR test for $H_0 : \theta_1 = \theta_2$ and $H_1 : \theta_1 \neq \theta_2$ under the assumption that $\theta_3 = \theta_4$. (10 points)

(5b) Find an LR test for $H_0 : \Delta\theta_3 = \theta_4$ and $H_1 : \Delta\theta_3 \neq \theta_4$, where $\Delta$ is a known constant. (10 points)

6. (20 points) Let $\boldsymbol{Y}$ denote an $n \times 1$ multivariate normal random vector, denoted by $\boldsymbol{Y} \sim N(\boldsymbol{X\beta}, \sigma^2 \boldsymbol{I}_n)$, where $\boldsymbol{X}$ is an $n \times p$ constant matrix with $rank(\boldsymbol{X}) = p$, $\boldsymbol{\beta}$ is a $p \times 1$ parameter vector, and $\boldsymbol{I}_n$ is the identity matrix of order $n$. Find an LR test for $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ and $H_1 : \boldsymbol{C\beta} \neq \boldsymbol{d}$, where $\boldsymbol{C}$ is an $r \times p$ full-row-rank constant matrix, $\boldsymbol{d}$ is an $r \times 1$ constant vector, and $r \leq p$.