1. (7%) Determine which of the following statements are true.

(A) The solution set of the matrix equation $X^2 + 3X + 2I_2 = O$ with the 2×2 matrix variable X is equal to $\{-I_2, -2I_2\}$.
(B) If the $n \times n$ matrix M is skew-symmetric, i.e., $M = -M^T$, then M is not invertible when n is an odd positive integer.
(C) Let T_A be the matrix transformation induced by the matrix A. For every square matrix A, $\text{Range}(T_A) \cap \text{Null}(T_A) = \{\mathbf{0}\}$.
(D) If A is an $m \times n$ matrix and E is an $n \times n$ elementary matrix, then Null A = Null AE.
(E) None of the above statements are true.

2. (7%) Which of the following subsets of $\mathcal{R}^n$ are a subspace.

(A) $S_1 = \left\{\begin{bmatrix} s \\ t \\ u \end{bmatrix} : s, t, u \in \mathcal{R},\ s + t + u = 1\right\}$.

(B) $S_2 = \left\{\begin{bmatrix} x \\ y \\ z \end{bmatrix} : x, y, z \in \mathcal{R},\ xyz = 0\right\}$.

(C) $S_3 = \left\{\mathbf{v} \in \mathcal{R}^4 : \begin{bmatrix} 1 & 2 & 3 & 4 \\ 8 & 7 & 6 & 5 \end{bmatrix}\mathbf{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\}$.

(D) $S_4 = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix}\right\}$.

(E) $S_5 = \text{Span}\left\{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}\right\} \cup \text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}\right\}$.

3. (7%)

$A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, $S = \left\{\begin{bmatrix} -3 \\ 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 4 \\ 5 \end{bmatrix}\right\}$. Which of the following statements are true?

(A) Let T_A be the matrix transformation induced by the matrix A. T_A is one-to-one.
(B) Let T_A be the matrix transformation induced by the matrix A. T_A is onto.
(C) The dimension of the range of $T_{A^T} = 2$.
(D) Null A = Span S.
(E) Let $\mathcal{B}_1$ be any basis for the range of T_{A^T} and $\mathcal{B}_2$ be any basis for Span S. Then $\mathcal{B}_1 \cup \mathcal{B}_2$ constitutes a basis for $\mathcal{R}^4$.

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4. (7%) Let A and B be $m \times n$ and $n \times k$ matrices respectively. Which of the following statements are true?

(A) $\text{Col}\, A$ is a subset of $\text{Col}\, AB$.
(B) $\dim(\text{Col}\, A) \leq \dim(\text{Col}\, AB)$.
(C) $\text{rank}\, A \geq \text{rank}\, AB$.
(D) Suppose that $k = n$ and B is an $n \times n$ invertible square matrix. Then $\text{Col}\, AB$ is a subset of $\text{Col}\, A$.
(E) None of the above statements are true.

5. (7%) Let Q be an $n \times n$ invertible matrix and P be an $n \times n$ matrix. Let R_1 and R_2 be the reduced row echelon form of Q and P respectively. Which of the following statements are true?

(A) R_1 is an identity matrix.
(B) If R_3 is also the reduced row echelon form of P, then R_3 must be equal to R_2.
(C) $\det(Q) = \det(R_1)$.
(D) Let $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a linearly independent set of vectors in $\mathcal{R}^n$. Then $\{P\mathbf{u}_1, P\mathbf{u}_2, \cdots, P\mathbf{u}_k\}$ is linearly independent.
(E) Suppose that $\{P\mathbf{u}_1, P\mathbf{u}_2, \cdots, P\mathbf{u}_k\}$ is linearly independent. It is necessary that $k \leq n$.

6. (7%) Determine which of the following statements are true?

(A) Let $\mathcal{B}$ be a basis of $\mathcal{R}^n$ and A be $n \times n$. If $\|A\mathbf{v}\| = \|\mathbf{v}\|$ for every vector $\mathbf{v} \in \mathcal{B}$, then A is orthogonal.
(B) If two square matrices have the same characteristic polynomials, then they are similar.
(C) In an inner-product space V, if $\langle \mathbf{u}, \mathbf{v}_i \rangle = \langle \mathbf{w}, \mathbf{v}_i \rangle$ for every vector $\mathbf{v}_i$ in a basis for V, then $\mathbf{u} = \mathbf{w}$.
(D) If two vectors are linearly independent and they are both eigenvectors of a symmetric matrix, then they are orthogonal.
(E) Let A be an $m \times n$ matrix and $W = \text{Col}\, A$. If P_W is the orthogonal projection matrix for W, then $A\mathbf{x} = P_W\mathbf{b}$ is consistent for every $\mathbf{b} \in \mathcal{R}^m$.

7. (8%) Let $T: P_2 \to P_2$ be defined by

$$T(p(x)) = 3p(0) + (-2p(1) + p'(0))x + p(2)x^2,$$

where

$$p'(0) = \left.\frac{dp(x)}{dx}\right|_{x=0}.$$

(A) T is a linear transformation.
(B) T is an isomorphism.
(C) The eigenvalues of T are 0 and -3.

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(D) A set of vectors consisting of the basis of each eigenspace for all eigenvalues of T constitutes a basis for $\mathcal{R}^3$.

(E) Let $\{1,\ x,\ x^2\}$ be a basis for P_2. Then any vector $\mathbf{v}$ in vector space P_2 can be uniquely represented as a linear combination of the vectors in $\{1,\ 1+2x,\ 1+2x+3x^2\}$.

For the following problems (8 to 17), assume: $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

8. (5%) The following differential equation can be classified as:

$$xy' + xy + 2y = x^2e^{-x}$$

(A) linear homogenous
(B) linear nonhomogenous
(C) nonlinear homogenous
(D) nonlinear nonhomogenous
(E) cannot determine

9. (5%) For the following differential equation, $x = -3$ is which type of point?

$$(x^2 + x - 6)y'' + (x + 3)y' + (x - 2)y = 0$$

(A) ordinary
(B) regular singular
(C) irregular singular
(D) none of these

10. (5%) Which is a solution of the following differential equation on interval (-5, 5)?

$$\frac{dy}{dx} = -\frac{x}{y}$$

(A) $x^2 + y^2 = 25,\ -5 < x < 5$

(B) $y = \sqrt{25 - x^2},\ -5 < x < 5$

(C) $y = -\sqrt{25 - x^2},\ -5 < x < 5$

(D) all of the above
(E) none of these

11. (5%) Which is a solution to the following differential equation?

$$y' + y = 0$$

(A) $y = 0$
(B) $y = e^{-x}$
(C) $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n$
(D) both A and B
(E) all of the above

12. (5%) Which is the general solution to the following differential equation?

$$y''' - 3y'' + 3y' - y = 0$$

Assume c_1, c_2, and c_3 are arbitrary constants.
(A) $y = c_1 e^x + c_2 e^x + c_3 e^x$
(B) $y = c_1 e^x + c_2 e^{x+1} + c_3 e^{x+2}$
(C) $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$
(D) $y = c_1 e^x + c_2 \ln x\ e^x + c_3 (\ln x)^2 e^x$
(E) none of these

13. (5%) Which is the general solution to the following differential equation?

$$x^2 y'' - 2xy' - 4y = 0$$

Assume c_1 and c_2 are arbitrary constants.
(A) $y = e^x [c_1 \cos(\sqrt{5}\, x) + c_2 \sin(\sqrt{5}\, x)]$
(B) $y = c_1 e^{-x} + c_2 e^{4x}$
(C) $y = x [c_1 \cos(\sqrt{5} \ln x) + c_2 \sin(\sqrt{5} \ln x)]$
(D) $y = c_1 x^{-1} + c_2 x^4$
(E) none of these

14. (5%) Which is the general solution to the following differential equation?

$$2y'' - 8y = 0$$

Assume c_1 and c_2 are arbitrary constants.
(A) $y = c_1 \cos(2x) + c_2 \cos(2x)$
(B) $y = c_1 \cosh(2x) + c_2 \cosh(2x)$
(C) $y = c_1 \cos(2x) + c_2 \sin(2x)$
(D) $y = c_1 \cosh(2x) + c_2 \sinh(2x)$
(E) none of these

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15. (5%) Which is the general solution to the following differential equation?

$$\frac{d^4y}{dx^4} + 2\frac{d^2y}{dx^2} + y = 0$$

Assume c_1, c_2, c_3, and c_4 are arbitrary constants.

(A) $y = c_1\cos(x) + c_2\sin(x) + c_3\cos(-x) + c_4\sin(-x)$
(B) $y = c_1\cos(x) + c_2\sin(x) + c_3\,x\cos(x) + c_4\,x\sin(x)$
(C) $y = c_1\cos(x) + c_2\sin(x) + c_3\,e^x\cos(x) + c_4\,e^x\sin(x)$
(D) $y = c_1\cos(x) + c_2\sin(x) + c_3\,e^{-x}\cos(x) + c_4\,e^{-x}\sin(x)$
(E) none of these

16. (5%) For the following differential equation on the interval $(0,\infty)$:

$$2xy'' + (1+x)y' + y = 0$$

which of the following equations are solutions?

(A) $y = \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{n}x^n$

(B) $y = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\,n!}x^{(n+1/2)}$

(C) $y = 1 + \sum_{n=1}^{\infty}\frac{(-1)^n}{1\cdot3\cdot5\cdot7\cdots(2n-1)}x^n$

(D) Both A and B
(E) Both B and C

17. (5%) Which is the general solution to the following differential equation?

$$y'' + y = 4x + 10\sin(x)$$

Assume c_1 and c_2 are arbitrary constants.

(A) $y = c_1\,cos(x) + c_2\,sin(x) + 4\,x$
(B) $y = c_1\cos(x) + c_2\sin(x) + 4\,x - 5\sin(x)$
(C) $y = c_1\cos(x) + c_2\sin(x) + 4\,x - 5\cos(x)$
(D) $y = c_1\cos(x) + c_2\sin(x) + 4\,x - 5\,x\sin(x)$
(E) $y = c_1\cos(x) + c_2\sin(x) + 4\,x - 5\,x\cos(x)$

試題隨卷繳回