- Any device with computer algebra system is prohibited during the exam.
- Answer all questions. Show all of your calculations or reasoning.

1. (10%) Consider the function $f(x) = \begin{cases} xe^{ax} + b & \text{if } x < -2 \\ c(x+1)^3 & \text{if } -2 \le x \le 0. \\ (\cos(x))^{\frac{1}{x^2}} & \text{if } x > 0 \end{cases}$

   It is known that :

   - $f(x)$ is continuous at $x = -2$,
   - $\lim_{x \to -\infty} f(x) = -\sqrt{e}$,
   - $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) + \frac{1}{2}$.

   Find the values of a, b and c.

2. Let $a > 0$ and set $f(x) = \frac{\ln(1 + a^x)}{x}$ for $x > 0$.

   (a) (10%) Explain why $f(x)$ is a non-increasing function. Deduce that for any $p \ge q > 0$,

   $$(1 + a^p)^q \le (1 + a^q)^p.$$

   (b) (5%) Which of $(e^{\frac{1}{2}} + \pi^{\frac{1}{2}})^2$ and $(e^{\frac{1}{3}} + \pi^{\frac{1}{3}})^3$ is larger? Explain your answer.

3. (10%) Let $a > 0$. Using the substitution $u = \frac{1}{x}$, evaluate $I = \int_{\frac{1}{\sqrt{3}}}^{\sqrt{3}} \frac{1}{(1 + x^a)(1 + x^2)} dx$.

4. In this question, we set $I = \int_0^1 x^x \, dx$.

   (a) (10%) Let m and n be two positive integers. Find $\int_0^1 x^m \cdot (\ln x)^n \, dx$ in terms of m and n.

   (b) (5%) Hence, find a sequence of rational numbers $\{a_n\}_{n=0}^{\infty}$ such that $I = \sum_{n=0}^{\infty} (-1)^n \cdot a_n$.

   (c) (5%) Find a rational number c such that $|I - c| < 10^{-3}$. Justify your choice of c.

5. Suppose $x = x(t)$ and $y = y(t)$. Consider the system of ordinary differential equations :

   $$\frac{dx}{dt} = \tan(x^2 + y), \qquad \frac{dy}{dt} = \tan(y^2 - x).$$

   (a) (5%) Approximate the right hand sides of each equation by a linear approximation at $(0,0)$.

   (b) (5%) Using (a), approximate a solution to the differential equations with initial conditions $x(0) = 0.1$, $y(0) = 0$.

6. (15%) Find the points on the surface $y^2 = 9 + xz$ that are closest to the origin.

7. Let $a > 0$ and R_a be the triangular region on xy-plane with vertices $(0,0)$, $(0,a)$ and $(a,0)$.

   (a) (10%) Let f be a continuous function on $[0,a]$. Show that

   $$\iint_{R_a} f(x+y) \, dA = \int_0^a u f(u) \, du.$$

   (b) (10%) Let U be the solid below the surface $z^2 = x + y$ and above the triangle R_1 on the xy-plane. Find

   $$\iiint_U e^{5z - z^5} \, dV.$$

試題隨卷繳回