1. (15 points)
Let $\{(X_n, d_n)\}$ be a sequence of metric spaces, and let $X = \prod X_n$. For $x, y \in X$, define

$$d(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \cdot \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}.$$

(a) Prove that d is a distance on X.
(b) (X, d) is a complete metric space if and only if each (X_n, d_n) is complete.

2. (15 points)
Determine the limit points of the sequence $\{\cos(n)\}_{n=1}^{\infty}$, and prove your result.

3. (15 points)
A set $S \subset \mathbb{R}^1$ is called compact if every open covering of S, there exists a finite subcovering that covers S. Use this definition to show that any finite closed interval $[a, b]$ is compact.

4. (15 points)
Let S be the sphere of radius A about the origin, and $F(x,y,z) = (x^2+y^2+z^2)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$. Use divergence theorem to compute $\int\int_S F \cdot n$, where n is the unit outward normal of S.

5. (15 points)
Suppose $f(x)$ is defined on $[-1, 1]$ and $f'''(x)$ is continuous. Prove that the series

$$\sum_{n=1}^{\infty} (n(f(1/n) - f(-1/n)) - 2f'(0))$$

is convergent.

6. (10 points)
Suppose $f(x) : [0,1] \to [0,1]$ and the set $G = \{(x, f(x)); x \in [0,1]\}$ is a closed set in $\mathbb{R}^2$. Is f continuous ? Justify your result.

7. (15 points)
Suppose $f \in C^1[0,1]$ and is a homeomorphism of $[0,1]$ onto itself. Prove that there exists a sequence of polynomials $\{P_n(x)\}$ such that P_n converges to f uniformly and each P_n is also a homeomorphism of $[0,1]$ onto itself.

試題隨卷繳回