1. Let $X_1, \ldots, X_n$ be a random sample from a population with pdf given by

$f(x|\theta) = \sqrt{\frac{\theta}{\pi}} exp(-\theta x^2), \ -\infty < x < \infty$ for some $\theta > 0$.

(a) (10 points) Please use the moment generating function (mgf) to show that the first two population moments are $E(X) = 0$ and $E(X^2) = 1/(2\theta)$, respectively.
(b) (5 points) Find the method of moment estimator for θ.
(c) (5 points) Identify a sufficient statistic for θ.
(d) (5 points) Find the maximum likelihood estimator (MLE) for θ.
(e) (5 points) If $Y = \sqrt{2\theta}X$ with $X \sim f(x|\theta)$, then find the distribution of Y.
(f) (5 points) Find the distribution of the random variable $S = \sum_{i=1}^{n}\left(\sqrt{2\theta}X_i\right)^2$.
(g) (5 points) Use (f) to show whether the MLE is an unbiased estimator. If not, please propose an **unbiased** estimator based on the MLE.
(h) (5 points) Comment on whether there exists another **unbiased** estimator with smaller variance than the estimator in (f).

2. (5 points) Let $X_1, \ldots, X_n$ be independent identically distributed random variables from an Exponential distribution with an unknown parameter $\theta > 0$. Consider the class of estimators, $T_n(c)$, of θ

$$\left\{T_n(c) = c\sum_{i=1}^{n} X_i \mid c > 0\right\}.$$

Determine the value of c that minimizes the mean square error (MSE). Note that the pdf of the exponential distribution is $f(x|\theta) = \frac{1}{\theta}e^{-x/\theta}$.

3. Two nominal random variables X and Y, each variable has two categories. If each individual in the population can be cross-classified according to whether X=1 or 2 and Y=1 or 2. Let joint probability $P(X = i, Y = j) = p_{ij}, i = 1, 2. and\ j = 1, 2$. Then, the marginal distribution of X is $P(X = i) = p_{i1} + p_{i2} = p_{i+},\ i = 1, 2$, and the marginal distribution of Y is $P(Y = j) = p_{1j} + p_{2j} = p_{+j}, j = 1, 2$.

(a) (5 points) If X and Y are independent, show that $\frac{p_{11}}{p_{+1}} = \frac{p_{12}}{p_{+2}}$ and $\frac{p_{11}}{p_{1+}} = \frac{p_{21}}{p_{2+}}$

When a sample is randomly collected from the population in a fixed time interval, let n_{ij} be the number of individual that X = i and Y = j. In this case, n_{ij} follow a Poisson distribution with pmf: $P(n_{ij}) = e^{-\mu_{ij}}\frac{\mu_{ij}^{n_{ij}}}{n_{ij}!}, i = 1, 2\ and\ j = 1, 2$.

To test H_0: $X\ and\ Y$ are independent vs. H_1: $not\ H_0$ (Hint: $xlog\left(\frac{x}{x_0}\right) \approx (x - x_0) + \frac{1}{2x_0}(x - x_0)^2$)

(b) (5 points) Write down the likelihood ratio(LR) test statistic and what distribution does it approximately follow?
(c) (5 points) Show that the LR test statistic is approximation to Pearson's Chi-square statistic.

When a sample is randomly collected from the population with a fixed size n, let n_{ij} be the number of individual that X=i and Y=j, $i = 1, 2\ and\ j = 1, 2$. In this case, $(n_{11}, n_{12}, n_{21}, n_{22})$ follow a multinomial distribution with pmf $P(n_{11}, n_{12}, n_{21}, n_{22}) = \frac{n!}{n_{11}!n_{12}!n_{21}!n_{22}!}p_{11}^{n_{11}}p_{12}^{n_{12}}p_{21}^{n_{21}}p_{22}^{n_{22}}$.

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To test $H_0: X\ and\ Y$ are independent vs. $H_1: not\ H_0$ (Hint: $x\log\left(\frac{x}{x_0}\right) \approx (x-x_0) + \frac{1}{2x_0}(x-x_0)^2$)

(d) (5 points) Write down the likelihood ratio(LR) test statistic and what distribution does it approximately follow?

(e) (5 points) Show that the LR test statistic is approximation to Pearson's Chi-square statistic.

When a sample with size n_1 is randomly collected from the population conditional on X=1 and let Y_1 be the number of individual that Y=1, then Y_1 will follow a binomial distribution B(n_1, p_1).

And another sample with size n_2 is randomly collected from the population conditional on X=2 and let Y_2 be the number of individual that Y=2, then Y_2 will follow a binomial distribution B(n_2, p_2).

To test $H_0: p_1 = p_2$ vs. $H_1: not\ H_0$ (Hint: $x\log\left(\frac{x}{x_0}\right) \approx (x-x_0) + \frac{1}{2x_0}(x-x_0)^2$)

(f) (5 points) Write down the likelihood ratio(LR) test statistic and what distribution does it approximately follow?

(g) (5 points) Show that the LR test statistic is approximation to Pearson's Chi-square statistic.

4. Let X be the number of calls received during any one hour, and follow a Poisson distribution with *pmf*: $P(X=x|\lambda) = \frac{\lambda^x}{x!}e^{-\lambda}, x = 0, 1, 2, \ldots$. To test $H_0: \lambda = 5$ $vs.$ $H_A: \lambda > 5$ and we know

	$x=0$	$x=1$	$x=2$	$x=3$	$x=4$	$x=5$
$P(X=x) = \frac{5^x}{x!}e^{-5}$	0.006	0.033	0.084	0.140	0.175	0.175

(a) (3 points) Write down the test statistic.

(b) (3 points) When significant level is set as 0.05, fine the rejection region of X.

(c) (3 points) When $X \in \{0, 1\}$, reject H_0. Find the probability of type I error.

(d) (3 points) Test $H_0: \lambda \le 5$ $vs.$ $H_A: \lambda > 5$ with the same significant level α given in Question(c), find the rejection region.

(e) (3 points) When $X \in \{0, 1\}$, reject H_0. Find the probability of type II error when $\lambda = 5^+$

試題隨卷繳回