- Unless otherwise specified, everything is over $\mathbb{R}$.
- The ordinary inner product of $\mathbb{R}^n$ is denoted by $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}}$.
- $\mathcal{S}_n$ is the space of $n \times n$ square matrices.
- $\mathcal{P}$ is the vector space of polynomials of one variable x with real coefficients.
- Dual space V^* of real vector space V is $\{\alpha \mid \alpha : V \to \mathbb{R}, \alpha \text{ is linear}\}$.

(1) [16%] $V \subset \mathbb{R}^4$ is a subspace span by $\vec{\mathbf{u}} = [1\ -4\ 8\ 3]^t$ and $\vec{\mathbf{v}} = [2\ -2\ 10\ 3]^t$. Define a linear transformation $T : V \to V$ by

$$\begin{aligned} T(\vec{\mathbf{u}}) &= 5\,\vec{\mathbf{u}} + 2\,\vec{\mathbf{v}} \\ T(\vec{\mathbf{v}}) &= 7\,\vec{\mathbf{u}} + \vec{\mathbf{v}} \end{aligned}$$

The induced inner product of V from $\mathbb{R}^4$ is defined by $<\vec{\mathbf{x}}, \vec{\mathbf{y}}> = \vec{\mathbf{x}} \cdot \vec{\mathbf{y}}$, $\vec{\mathbf{x}}, \vec{\mathbf{y}} \in V$. Is T self-adjoint with respect to $< , >$? Demonstrate your answer.

(2) [16%] $\mathcal{P}_3 \equiv \{f(x) \in \mathcal{P} \mid \deg(f(x)) \leq 3\}$. Let $\mathcal{P}_3^*$ be the dual space of $\mathcal{P}_3$. For any $a \in \mathbb{R}$, define $\widehat{a} \in \mathcal{P}_3^*$ by $\widehat{a}(f(x)) = f(a)$ and $\mathrm{d}\hat{a} \in \mathcal{P}_3^*$ by $\mathrm{d}\hat{a}(f(x)) = f'(a)$.

a. Find the basis $\phi_{-1}(x)$, $\phi_0(x)$, $\phi_d(x)$, $\phi_1(x)$ of $\mathcal{P}_3$ such that $\widehat{-1}$, $\widehat{0}$, $\mathrm{d}\hat{0}$, $\widehat{1}$ are their corresponding dual basis.

b. Define $I \in \mathcal{P}_3^*$ by $I(f(x)) = \int_{-1}^{1} f(x)dx$. Find $\alpha, \beta, \gamma, \epsilon \in \mathbb{R}$ such that

$$I = \alpha\,\widehat{-1} + \beta\,\widehat{0} + \gamma\,\mathrm{d}\hat{0} + \epsilon\,\widehat{1}$$

c. If there is $f(x) \in \mathcal{P}_3$ such that $f(-1) = -2$, $f(0) = 2$, $f'(0) = \pi$, $f(1) = -6$, evaluate $\int_{-1}^{1} f(x)dx$.

(3) [16%] $\Gamma = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} \in \mathcal{S}_n$. $\mathcal{C}_n = \{X \mid X\Gamma = \Gamma X\}$ is a subspace of $\mathcal{S}_n$.

Determine $\dim \mathcal{C}_n$ and find a basis of $\mathcal{C}_n$.

(4) [16%] $A \in \mathcal{S}_n$. Define m_{ij} to be the determinant of the submatrix formed by deleting the i-th row and j-th column of A. Define the classical adjont matrix $\operatorname{adj} A = [(-1)^{i+j} m_{ji}]$. Suppose A is not invertible, show that rank of $\operatorname{adj} A$ is ≤ 1. When is the rank of $\operatorname{adj} A = 1$?

(5) [16%] If $A = [a_{ij}] \in \mathcal{S}_n$ is positive definite, show that $\det A \leq a_{11} a_{22} \cdots a_{nn}$.

(6) [20%] $A \in \mathcal{S}_n(\mathbb{C})$. Over $\mathbb{C}$, show the following two statements are equivalent.

a. The characteristic polynomial of A is equal to minimal polynomial of A.

b. For any $X \in \mathcal{S}_n(\mathbb{C})$ satisfies $XA = AX$, X is a polynomial of A.

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