1. (15 points)
Suppose $f:(a,b)\to\mathbb{R}$ satisfies that for all $x,y\in(a,b)$ we have $|f(x)-f(y)|\le M|x-y|^c$ for some fixed positive constants M and c.
(a) Show that f is uniformly continuous.
(b) Show that f extends uniquely to a continuous function defined on $[a,b]$.

2. (15 points)
Let $f_n(x)=e^{-nx}$. (a) Show that $f_n(x)$ is not uniformly convergent to 0 on $(0,\infty)$. (b) $f_n(x)$ is uniformly convergent to 0 on $[a,\infty)$ for any $a>0$.

3. (20 points)
Let $f(x,y)=2x^4-3yx^2+y^2$.
(a) Show that $(0,0)$ is a degenerate critical point of f (i.e the Hessian of f is singular).
(b) Given any $(a,b)\neq(0,0)$, show that the function $g(t)=f(at,bt)$ has a local minimum at the origin, but f does not have a local minimum at the origin.

4. (10 points)
Let $f(x,y)=e^{x^2+xy+y^2}$ and $a=(0,0)$. Find the 4-th order Taylor polynomial of f at a.

5. (20 points)
Consider the metric space (X,d), where X is the set of all real valued continuous functions defined on $[0,1]$, and for $f,g\in X$, let $d(f,g)=\max_{x\in[0,1]}|f(x)-g(x)|$. Let $K=\{\cos(nx)\}_{n=1}^{\infty}\subset X$,
(a) Is K bounded ?
(b) Is K compact ?
(c) Is K closed ? Justify all your results.

6. (20 points)
Given a function $f:\mathbb{R}\to\mathbb{R}$. Show that the following set E is at most countable.

$$E=\{a\in\mathbb{R};\lim_{x\to a}f(x)\text{ exists and }\lim_{x\to a}f(x)\neq f(a)\}.$$

試題隨卷繳回