題號： 267　　國立臺灣大學 109 學年度碩士班招生考試試題
科目： 工程數學(G)
節次： 6

1. (30%). Let $w=i$ where $i^2=-1$. Set $\mathbf{F}=\begin{pmatrix}1&1&1&1\\1&w&w^2&w^3\\1&w^2&w^4&w^6\\1&w^3&w^6&w^9\end{pmatrix}$, $\mathbf{A}=\begin{pmatrix}d&-1&0&-1\\-1&d&-1&0\\0&-1&d&-1\\-1&0&-1&d\end{pmatrix}$ and $\mathbf{x}=\begin{pmatrix}1\\w^j\\w^{2j}\\w^{3j}\end{pmatrix}$, where d is a real number and j is an integer.

(a). (15%). Determine $\mathbf{F}^{-1}$, the inverse of $\mathbf{F}$. (Hint: computing $\mathbf{F}\overline{\mathbf{F}}$ where $\overline{\mathbf{F}}$ is the complex conjugate of $\mathbf{F}$)

(b). (15%). Find the real eigenvalues and eigenvectors of $\mathbf{A}$. (Hint, first compute $\mathbf{Ax}$; second simplify $\mathbf{Ax}$ and see if there exists λ such that $\mathbf{Ax}=\lambda\mathbf{x}$; finally set j=0, 1, 2 and 3 in the result of $\mathbf{Ax}$).

2. (35%). The "backwards heat equation" in one spatial dimension is shown below:

$$\frac{\partial u}{\partial t}=-\frac{\partial^2 u}{\partial x^2},\qquad (t,x)\in[0,1]\times[0,1].$$

(a). (10%). Use the separation of variable method to find all solutions to the "backwards heat equation" of the form $u(t,x)=T(t)X(x)$ that satisfy the boundary conditions $u(t,0)=0$ and $u(t,1)=0$ for $t\in[0,1]$.

(b). (15%). Discuss the behavior of the solution $u(t,x)$ to the following initial/boundary conditions:

$$u(0,x)=f(x)=\sin(n\pi x) \text{ where } n>0 \text{ is the integer for } x\in[0,1],$$
$$u(t,0)=0 \text{ and } u(t,1)=0 \text{ for } t\in[0,1].$$

(b1). (10%). Solve the "backwards heat equation."

(b2). (5%). When $t=1$, find $\max\limits_{x\in[0,1]} u(x)$.

(c). (10%). Solve the ordinary heat equation $\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}$ with the initial/boundary conditions:

$$u(0,x)=f(x)=\sin(n\pi x) \text{ where } n>0 \text{ is the integer for } x\in[0,1],$$
$$u(t,0)=0 \text{ and } u(t,1)=0 \text{ for } t\in[0,1].$$

How do the Fourier modes of the solution $u(t,x)$ from (b1) and the Fourier modes of the solution for the ordinary heat equation change in time?

3. (25%). Answer the following questions:

(a). (15%). Solve $\frac{dP}{dt}=P(a-bP)$, $P(0)=P_0$ and $P_0\neq\frac{a}{b}$ where a and b are constants.

(b). (5%). Continuing from (a), find the value(s) of P where $P(t)$ has point(s) of inflection. Recall that points of inflection of a function can occur where its second derivative is zero.

(c). (5%). Solve $\frac{dP}{dt}=P(a-b\ lnP)$ where a and b are constants.

4. (10%). Consider the following differential equation:

$$xy''=y'+(y')^3.$$

(a). (5%). Show that substituting $u=y'$ leads to a Bernoulli equation in terms of u.

(b). (5%). Based on (a), solve this equation.

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