※ 注意：請於試卷內之「非選擇題作答區」標明題號依序作答。

Problem 1. (18%)

Given the following matrices:

$$\mathbf{A}=\begin{bmatrix}1 & -1 & -1\\ -1 & 1 & -1\\ -1 & -1 & 1\end{bmatrix}\quad \mathbf{P}=\begin{bmatrix}1 & -2 & 0\\ 1 & 1 & 1\\ 1 & 1 & -1\end{bmatrix}\quad \mathbf{Q}=\begin{bmatrix}0 & \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}}\end{bmatrix}$$

(a) Find the eigenvalues and three orthonormal eigenvectors of $\mathbf{A}$. (10%)

(b) Find $\mathbf{P}^{-1}\mathbf{AP}$, where $\mathbf{P}^{-1}$ is the inverse of $\mathbf{P}$. (4%)

(c) Find $\mathbf{Q}^T\mathbf{AQ}$, where $\mathbf{Q}^T$ is the transpose of $\mathbf{Q}$. (4%)

Problem 2. (15%)

Given a vector field $\mathbf{F}(x,y,z)=-y\,\mathbf{i}+x\,\mathbf{j}-xyz\,\mathbf{k}$ and a surface $S:\ z=\sqrt{x^2+y^2}$ for $x^2+y^2\le 4$ which is bounded by the curve $C:\{x^2+y^2=4,\ z=2\}$.

(a) What is the curl of $\mathbf{F}$? (5%)

(b) Find the value of $\iint_S (\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,ds$, where $\mathbf{n}$ is a unit normal to S in the direction of the orientation of S. (10%)

Problem 3. (11%)

Expand $f(x,y)=e^{x+y}$, $0\le x\le 4$, $0\le y\le 2$, in a double Fourier sine series, given the following integral.

$$\int e^{\alpha x}\sin\beta x\,dx=\frac{e^{\alpha x}(\alpha\sin\beta x-\beta\cos\beta x)}{\alpha^2+\beta^2}+C$$

Problem 4. (11%)

Pick the correct answer for the following integral. You don't need to provide the calculation procedures.

$$\int_{-\infty}^{\infty} \frac{\sin\pi x \cos\pi x}{2x^2 - x} dx$$

(A) 0

(B) π

(C) $-\pi$

(D) $\pi/2$

(E) $-\pi/2$

Problem 5. (11%)

A Sturm-Liouville system having the form:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + [q(x) + \lambda r(x)]y = 0$$

$$\alpha_1 y(a) + \alpha_2 y'(a) = 0$$

$$\beta_1 y(b) + \beta_2 y'(b) = 0$$

$$a \le x \le b$$

Show that the eigenvalues of Sturm-Liouville system are real. Assume $p(x)$, $q(x)$, $r(x)$, α_1, α_2, β_1, β_2 are real, while λ and y may be complex.

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Problem 6. (10%)

Please derive the corresponding 'weight functions' of the following two ordinary differential equations by putting them into their self-adjoint forms: (5% each)

(i) $\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2ny=0$,

(ii) $\left(1-x^2\right)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+n^2y=0$.

Note that polynomial particular solutions to the above two equations can be generated when n equals to non-negative integers.

Problem 7. (24%)

(a) (12%) Please 'write down' the two families of characteristic lines of the wave equation, namely,

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2} \quad \text{for} \quad -\infty<x<\infty.$$

(b) (12%) Using the characteristic lines or coordinates obtained from part (a), please perform a change of variables and derive the d'Alembert's formula for the wave equation,

$$\frac{\partial^2 u}{\partial t^2}=c^2\frac{\partial^2 u}{\partial x^2},$$

with the initial conditions of $u(x,t=0)=\phi(x)$ and $\frac{\partial u}{\partial t}(x,t=0)=\psi(x)$ for $-\infty<x<\infty$, $t\geq 0$.