國立臺灣大學 104 學年度碩士班招生考試試題

1. (15%) Find the general solutions of the following ODE's.
   (a) $y' = y\, e^x$
   (b) $y' + y = x/y$
   (c) $x^2 y'' + 4xy' - 4y = 0$

2. (a) (7%) Find the Laplace transform for $y(t) = t^2 e^{-at}$.

   (b) (8%) If the Laplace transform of $y(t)$ is $Y(s) = \frac{s^3}{s^4+4}$, please find

   (i) the Laplace transform of $\frac{dy}{dt}$ (ii) the value of $\frac{dy}{dt}$ at t=0.

3. (10%) Solve the differential equation $y'' + y' + 2\, xy = 0$ by series solution:
   (a) Assuming $y(x) = \sum_{m=0}^{\infty} a_m x^m$, find the recurrence relation for a_m.
   (b) Select two separate sets of $[a_0, a_1] = [0, 1]$ and $[1, 0]$, find the series solutions for each set up to 4 terms.

4. (10%) The definition of Bessel function of order n is

$$J_n(x) = x^n \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{2^{2m+n} m!\,(n+m)!}$$

   (a) Show that $\int xJ_0(x)dx = xJ_1(x)$.
   (b) Show that $J_0(x)$ has a bounded value everywhere.

5. You will be solving the following partial differential equation (PDE) problem using the "Method of Separation of Variables":

$$u_t = u_{xx} \ \text{ for } \ 0 \le x \le \pi \ \text{ and } \ t > 0$$
$$\text{B.C.s}: \ u_x(0,t) = u_x(\pi,t) = 0 \ \text{ for } \ t > 0$$
$$\text{I.C.}: \ u(x,0) = \cos^2(x) \ \text{ for } \ 0 \le x \le \pi$$

(4%) (a) Why can the "Method of Separation of Variables" be applied to this problem?
(6%) (b) What are the eigenvalues and eigenfunctions associated with this problem?
(10%) (c) Please find the solution of this PDE problem

見背面

6. If a vector field $\underline{F} = (2xy - y^4 + 3)\underline{i} + (x^2 - 4xy^3)\underline{j}$, C: the path connecting (0,0) and (2,1).

(5%) (a) Please show the reason why $\underline{F}$ is a conservative vector field.

(5%) (b) If $\underline{F}$ is a conservative vector field, there exists a (scalar) potential function ϕ. Please also show the relationship between $\underline{F}$ and ϕ.

(5%) (c) Please calculate ϕ and $\int_C \underline{F} \bullet d\underline{r}$.

7. You are given the following matrix A:

$$A = \begin{bmatrix} 5 & 4 \\ -6 & -5 \end{bmatrix}$$

(4%) (a) Find the eigenvalues and eigenvectors of the matrix A

(4%) (b) If A is similar to D (a diagonal matrix with eigenvalues as the diagonal components), what are the transition matrix P and its inverse P^{-1} ?

(7%) (c) Use the results from (a) and (b) to solve the following system of ODEs:

$$\frac{dx_1}{dt} = 5x_1 + 4x_2 + e^{2t}$$

$$\frac{dx_2}{dt} = -6x_1 - 5x_2 + 2e^{2t} + t$$

試題隨卷繳回