國立臺灣大學 104 學年度碩士班招生考試試題

※ 注意：請於試卷上「非選擇題作答區」內依序作答，並應註明作答之大題及其題號。

1. (10%) Given $L^{-1}\left\{\frac{1}{\sqrt{s^2+1}}\right\}=J_0(t)$ and $L^{-1}\left\{\frac{s}{\sqrt{s^2+1}}-1\right\}=-J_1(t)$

   Solve the nonlinear integral equation

   $$x(t)-\frac{1}{2}\int_0^t x(t-\tau)x(\tau)\,d\tau=\frac{1}{2}\sin(t)$$

   by Laplace transformation. Express your answer in terms of $J_n(t)$ and $\delta(t)$.

2. (10%) A boundary value problem having the form

   $$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right]+\left[q(x)+\lambda r(x)\right]y=0 \quad \text{for } a\le x\le b;$$

   $$\alpha_1 y(a)+\alpha_2 y'(a)=0 \quad \text{and} \quad \beta_1 y(b)+\beta_2 y'(b)=0.$$

   Show that the eigenfunctions y_1 and y_2 belonging to two different eigenvalues λ_1 and λ_2 are orthogonal with respect to $r(x)$ in (a,b).

3. (15%) In calculus, the curvature of a curve $y=f(x)$ is defined as

   $$\kappa=\frac{y''}{\left[1+(y')^2\right]^{3/2}}$$

   Find $y=f(x)$ for which $\kappa=1$. For simplicity, ignore constants of integration.

4. (35%) Consider an elastic string whose displacement function satisfies

   $$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2} \quad \text{for } t>0.$$

   (a) If the string is infinitely long and the initial conditions are

   $$y(x,0)=0 \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0)=\delta(x),$$

   determine the Fourier transform of the displacement function and the maximum displacement value.

   (b) If the string has length L with conditions:

   $$y(0,t)=0 \quad \text{and} \quad y(L,t)=0;$$

   $$y(x,0)=\sin\left(\frac{2\pi x}{L}\right) \quad \text{and} \quad \frac{\partial y}{\partial t}(x,0)=\sin\left(\frac{5\pi x}{L}\right),$$

   solve for the displacement function.

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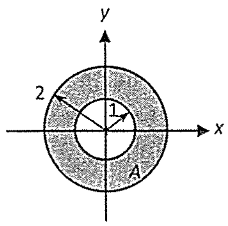
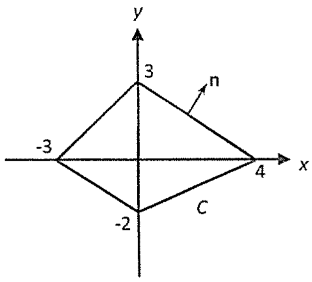
5. (15%) Let $\phi(x,y) = x + \ln(x^2 + y^2)$ and $\mathbf{v}(x,y) = (x^2 \cos y)\mathbf{i} + (y^2 \sin x)\mathbf{j}$ be 2-D scalar and vector functions, respectively; and let $\mathbf{u}(x,y) = \nabla\phi + \nabla \times \mathbf{v}$. Write down the answers to the following questions. **(Derivations are not required.)**

(a) Evaluate the surface integral $\iint_A \nabla \cdot \mathbf{u}\, dA$ over an annulus region between two concentric circles as shown below.



(b) Evaluate the line integral $\oint_C \mathbf{n} \cdot \mathbf{u}\, d\ell$ over a closed curve C as shown below. (with $\mathbf{n}$ denoting a unit vector normal to the curve C)



(c) Evaluate the line integral $\oint_C \mathbf{u} \cdot d\mathbf{r}$ along a circle C of radius 1 centered at the origin. (where $d\mathbf{r}$ is a displacement vector along the curve C)

6. (15%) Let $z = x + iy$ denote the complex variable, $\bar{z}$ be the complex conjugate of z, and $f(z)$ a complex function. Answer the following questions. **(Derivations are not required.)**

(a) If the real part of an analytic function $f(z)$ is $xe^x \cos y - ye^x \sin y$, what is the imaginary part of $f(z)$?

(b) Let $f(z)$ be analytic on and inside the unit circle $|z| = 1$. If on the circle $|z| = 1$,
$f(e^{i\theta}) = \dfrac{-3 + i4\sin\theta}{5 + 4\cos\theta}$, what is the expression of the function $f(z)$?

(c) Evaluate the complex integral $\oint_C z^2 \sin\bar{z}\, dz$ over the closed contour C defined by $|z| = 1$.

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